A BACKGROUND TO
PRIMARY SCHOOL
MATHEMATICS

A BACKGROUND TO PRIMARY SCHOOL MATHEMATICS

BY

L. D. ADAMS

LATELY H.M. INSPECTOR OF SCHOOLS

OXFORD UNIVERSITY PRESS
1953

Oxford University Press, Amen House, London E.C.4
GLASGOW NEW YORK TORONTO MELBOURNE WELLINGTON
BOMBAY CALCUTTA MADRAS KARACHI CAPE TOWN IBADAN
Geoffrey Cumberlege, Publisher to the University

510
A.L16 B

LAMA IQBAL LIBRARY
093

JAMMU & KASHMIR UNIVERSITY
LIBRARY
No. 13093
Date 22-8-55
SRINAGAR

ST82

CHECKED

PRINTED IN GREAT BRITAIN

PREFACE

THIS book is not intended as a book of child psychology or as a treatise on elementary mathematics. It is, so far as possible, an account of personal experience and opinion based on many years of observation of children of primary school age. My opinions of necessity owe a great deal to persons I have met who are interested in the same problem and to books, but it is difficult to identify and impossible to acknowledge fully help received from such a multitude of sources. Many of the ideas initiated in reading or discussion have become second nature through testing them in experience. I can do no more than disclaim originality and take full responsibility for errors of judgement.

To the little book from which quotation is made in some chapter headings I owe much. Several of the books of Mary Everest Boole have been on my shelf since my student days. I have taken down the book from which I quote, *Preparation of the Child for Science*, at intervals over the years and I have never failed to enjoy re-reading it and searching out its pearls of wisdom. It is humbling to think how little we advance in fifty years.

I also owe a great deal to the discussions in which I have taken part at meetings of the primary school sub-committee of the Mathematical Association. I have tried to keep the writing of these pages a record of personal experience, but development of opinion from discussion is inevitable, and sub-committee meetings have been most stimulating.

Some examples are given of children's work. These have been given to me by teachers and by students in training who know that I am interested in the mathematical development of children. I am very grateful. Except for quite recent additions to my collection, examples are from work shown in an exhibition at the Mathematical Association meeting in 1951. The examples are necessarily limited in type. It is little use showing computation only, except perhaps when it is wrong, because it gives little indication of the worker's habits and attitude. Work accompanied by verbal recording and by pictures on the other hand is rather more revealing; it is work of this kind which is quoted. Wherever the actual form as well as the substance of the work seemed of interest a copy of the original has been made. Figures so copied are marked with an asterisk in the list of illustrations. My very best thanks go to Mr. Ewart Uncles who has so sensitively and skilfully caught the childlike characteristics of the originals in preparing copies suitable for reproduction by line blocks. I should also like to thank members of the staff of the Oxford University Press, who have taken so much trouble with the other figures and with the arrangement of the text.

Many stories are told about children. Most of these are not meticulous records. Exceptions are the stories of Terry, Julia, and Rosie and Billy, which are given, not quite in full, in the form in which they were recorded from close observation. Special thanks are due to the recorders, all experienced teachers, for permission to use them. All the stories are founded on observation or reliable hearsay; some of them are composite; many of them have been told many times and have become standardized in the telling; a few are idealized in that lost opportunities have been recaptured and taken to an imaginary conclusion in line with general experience. Most of the children and their teachers appear incognito, but stories under the same name are of the same child on different occasions. The significance of details in the stories has often only become apparent in retelling them; analysis of these details owes much to teachers' comments.

Where schools or classes are described, the descriptions are generally composite pictures and where reference is made to 'a school' in order to give the background of an individual, any details given must be read as referring to types and systems rather than to the picture of a particular class containing a particular child. The few exceptions are items specially contributed to fill a gap in the illustration of this book.

In approaching children I have sometimes allowed myself the dangerous question 'What did you do?' or worse 'How did you do it?' Whenever possible, however, I have tried to create situations which called for spontaneous comment from a child or to make comment myself which stimulated response without a direct question. Whenever I have re-created situations which have proved fruitful elsewhere, I have tried to standardize my own part in the proceedings. Readers who try to copy such situations are warned that they may have to create their own standards of response by frequent trials, because of the inevitable personal differences in presenting the situations. Generally the minimum of suggestion compatible with making the circumstances clear has been aimed at.

To the children themselves, Jill, Linda, Mary, and countless others, to their parents, aunts, and teachers, I owe my thanks not only for help but also for many happy hours.

To the stimulation and advice of the many friends with whom I have shared an interest both in how children learn and in our human heritage of mathematics I give most grateful acknowledgement. Some of these friends are beyond the reach of thanks, but to them all I offer this book in tribute to their generosity and to our common interest.

L. D. A.

BRISTOL, 1953

CONTENTS

INTRODUCTION 1

Scope and arrangement of book—Variety of interests and aptitudes of children—Two aspects of learning mathematics—To whom the book is addressed.

I. FIRST IDEAS OF NUMBER 3

Differences in children due to environment and experience—The entrant to school—Number names and the beginning of counting—Meanings for number names—First uses of number relations—Using environment and apparatus.

II. HOW NUMBERS ARE WRITTEN 16

Learning from environment: (*a*) figures, (*b*) written numbers over 9—Learning how large numbers are written—Meanings of large numbers common in environment—The connexion between counting in groups and writing numbers in tens and units—Apparatus.

III. MONEY AND THE BEGINNINGS OF MEASURING 24

Approach through environment.

Money—Shopping—Coins—Counting money.

Measuring—Fundamental experience of materials—Gulf between child and adult—Experiences with various materials—The beginnings of measuring—Length, idea of a unit, tools—Approach to measuring other quantities—Some special types of equipment—Teacher's part, especially in relation to vocabulary.

IV. TRANSITION: A CRITICAL DIGRESSION 41

What comes after learning to count—Abrupt introductions to symbolism and reckoning—Examples of what 'little sums' mean to children of 6, 7, and 8 years of age—A plea for reconsideration.

V. PRACTICAL BACKGROUND TO ARITHMETICAL PROCESSES 53

(*a*) And-ing, Comparing, Taking Away.

Experience of the processes—Complementary addition—Use of signs, a late stage—Filling out experience—Some difficult examples of subtraction.

(*b*) EQUAL GROUPINGS
Multiplication tables on inadequate experience—Counting equal groups—Relation to work with measures—Highlights in experience—Approach to inverse uses—Unfortunate isolation of arithmetic.

VI. NUMBER ACCOMPLISHMENTS: TABLES 71
Tables, the foundations of reckoning and the outcome of experience—Link with experience twofold—Introduction of symbols as meaning becomes clear—The time for systematization—Nought—Addition-subtraction tables—Occurrence of items—Notational extension—Multiplication-division tables—Knowledge of sequences—Outstanding items—How numbers behave—Patterns—Building up a knowledge of numbers—Notational extension.

VII. MEANS OF LEARNING: A SUMMARY 84
The overlap between Infant and Junior stages—Means of expression—Contrasting ways of managing classes with examples of children's work—Special problems of the Junior stage.

VIII. WRITTEN RECKONING 101
Differences of method—Readiness for learning—Sanction of use for the process as a whole and for its elements—Possibilities of experimental approach—Getting reckoning right—Addition, subtraction, multiplication, and division analysed—Compound quantities, similar in principle—Short methods.

IX. MONEY, WEIGHTS, AND MEASURES 118
An early and realistic approach instead of formal reckoning.
MONEY—Shopping—Mental and written work—Postponement of formal rules.
MEASURES—Weight, apparatus and range of work—Liquid measure, homely examples—Length, importance in mathematics, small and large measures, reckoning—Time, telling the time, experience and reckoning—Other measures, area, volume, speed.

X. FRACTIONS 136
Gradual introduction—Notation in everyday use—Relation to measures—Descriptive uses of notation—Experience of equivalence of fractions—Calculations from first principles—Multiplying and dividing a later stage—The bar notation, fractions and sharing, ratio and measuring—Decimal fractions seldom in young child's experience—Percentage.

XI. DRAWING, MODEL MAKING, DIAGRAMS 152
Traditional teaching of geometrical techniques and handwork—Paper folding and symmetry—Model making—Direction, angles, outdoor measuring—Moving parts and machines—Pictures, plans, maps, and diagrams—Particular examples of graphs and diagrams—An approach to measuring area—The syllabus.

XII. MISCELLANEOUS OPPORTUNITIES 170
Wide range of children's interest—An up-to-date example—Teacher's use of special knowledge, technical, statistical, &c.—New ideas and notations, directed number, formulation, conventions—Pure number interests, puzzles—History of mathematics—Books as a source of mathematical interest and information.

CONCLUSION: STANDARDS 182
Quality and range of achievement—Reckoning right not enough—The nature of accuracy in reckoning and matters of information and in practical work—Speed of working—The problems of the class teacher—The pressure of external tests—Mathematics and the growth of the whole child.

LIST OF ILLUSTRATIONS

1. Story illustration; ships to be counted 7
2. Page from a news book—Making and counting 9
3. Page from a news book—Numbers in a description 10
4. Page from a record of things made—Numbers and measures in a description 10
5. Number-pattern cards showing even numbers 12
6. Number-pattern cards showing two odd numbers making an even number 12
7. Gate wrongly closed 12
8. Number-pattern cards showing fourteen built up in two ways 13
9. Domino stair—Systematic arrangement, first method 14
10. Domino stair converted into a rectangle 15
*11. Self-set problem—Notation difficulty 18
*12. Early illustrated shopping record—'Costs of loaves' 25
13. Pages from a news book—Recording and comparing heights 32
*14. Self-set illustrated problem—'Eating nuts'—*subtraction* 44
*15. Self-set illustrated problem—'Wheels and bicycles'—*division* 44
*16. Self-set illustrated problem—'Cars and wheels'—*addition and multiplication* 46
*17. Self-set illustrated problems—'Wheels and cars' (pictures only)—'Feet and people'—*division* 46
*18. Self-set problems—'Desks and people'—*adding and doubling*—'Paws and cats'—*division with answer as divisor* 47
*19. Self-set illustrated problem—'Wheels and cars' (diagram only) 48
*20. Self-set problem with large numbers—'Dogs, cats, and legs'—*two-part problem with addition and multiplication* 48
*21. Self-set problems with small numbers—'Wheels and cars'—*addition or multiplication*—'Giving away ice cream'—*subtraction* 48
*22. Self-set problem with money—Comparing possessions—crude method 49
*23. Self-set illustrated problem—'Dogs and cats'—*very simple addition* 49
*24. Self-set problems with pictures—'Totalling people'—*adding tens*—'Cost of tickets'—*multiplying money* 50

25. Self-set illustrated problem—'Fathers with and without hats'——addition and subtraction forms of statement* (very backward child) 51

26. Domino stair. Systematic arrangement, second method 54

27. Scoring by columns of beads 57

28. Beads showing (*a*) comparison by difference, (*b*) taking away 58

29. Sharing liquid into cups 66

30. Number square (tens) and pattern of the tables of 3, 6, and 9 in the square 80

31. Patterns with 10 and 12 counters 82

*32. 'Figures and cars' (Terry) 86

*33. 'On the Farm'—Combined free writing, drawing, and reckoning 93

*34. 'Measures Remembered'—A free pictorial record 99

35. Experimental approach to reckoning—Seven fourteens 112

36. Cards with buttons—*practical exercise leading to equivalence of fractions* 141

37. Diagrams of fractions and sharing 145

*38. Diagrams made to show percentages from school attendance figures 151

39. A method of folding paper to make a right angle 153

40. 'Ghost' signature—*Symmetry* 154

41. Rectangle divided up by its diagonals 155

42. Sector of a circle as the 'net' of a cone—Drawing a border pattern 155

43. 'Net' of a rectangular tray 156

*44. 'Cows and calf'—Showing good relative sizes (young child) 159

*45. 'Lollipop and child'—Showing 'emotional' instead of actual scale (young child) 160

*46. Enlargement of a picture by enlargement of squares 161

*47. Picture graphs (isotypes) (child of 10 years) 163

48. Making a square from a strip 166

49. Patterns with coloured squares and half squares—*comparison of areas* 167

*50. Covering rectangles with squares 168

51. Area of an L-shaped figure 169

52. Economical ways of finding averages 173
*53. 'The Alphabet and the Morse Code'—*frequency investigation* (children of 10 years) 175
54. Diagram showing error made in measuring 184

Figures marked with an asterisk are copies of children's original work

INTRODUCTION

THE mathematical range of this book does not go beyond the scope of an able child of 11 years of age except for passing references which are not essential to the argument. The ordering of the chapters follows no very clearly marked line of development in dealing either with a child's mental powers or with subject-matter. Development of mental powers has been discarded as a guide to sequence partly because observations are based on things as they are rather than on ordered experiment. To take subject-matter in logical order seemed on the other hand too great a concession to the adult point of view. Indeed, the separation of this mathematical aspect of a child's growth from the rest of his development is concession enough to the adult outlook. A synopsis of each chapter is given in order to compensate for this lack of system in arrangement.

The variety of the interests and aptitudes of children has led to the inclusion of material relevant to a wide range of age in almost every chapter. Roughly, however, the earlier chapters seldom contain comment on the work of children over 9 years of age and references to infants become fewer in later chapters. The teacher of backward children, however, will realize that much of what is said about mathematics itself at the infant stage will apply to older pupils who have failed in learning from their early experience, and that these children need to explore in the same simple way new fields of experience suited to their ages and interests.

There is little systematic in the treatment given to differences of ability, though the fact of their existence is a main theme of the argument. Ideal criteria of readiness for the teaching of particular items can only be applied to individuals. Some compromise is necessary for the class teacher, but disregard for the nature of readiness in favour of arbitrary subject syllabuses must leave many children's minds ill trained or even in confusion.

Another underlying theme of the argument is the delicate nature of the teacher's task in keeping at this early stage a balance between two aspects of mathematical learning, namely a growth of understanding on the one hand and the development of habits of thought on the other. In speaking of the skills of arithmetic it is this second aspect only which is emphasized. Skill and understanding are not independent, because skill at one level may help advance into higher fields of thought, but the acquisition of skill and a state of contemplation require different settings and are out of phase in a mental rhythm. If a child is trained in the one while being given too little opportunity for the other, the skill he acquires may be ill founded and unconnected with anything he understands. Contemplation is no inactive state, but the teacher's part in fostering the growth of understanding is often indirect, and it is difficult to write about because it demands of the teacher restraint as well as action. Again most of the mathematical skills acquired in primary school life are skills in dealing with number, but the foundations of understanding are being laid from a very early age over a very wide field. The evidence of growth is often in the transitory form of speech or simply in the way a child sets about doing something which would certainly not appear as mathematics on any time-table.

No theorizing, however, can take the place of first-hand observation of children. It is hoped that some account of the observation and practice of others may suggest what there is to look for and how even difficulties can be used to advantage.

The book is addressed in the first place to teachers and intending teachers in all types of school, to primary school teachers because it is a part of their problem which is discussed, to teachers in secondary schools because education is a continuous cycle and the primary school children of yesterday are the secondary school children of today and the parents and educators of tomorrow. It is hoped too that parents interested in their children's education may find in the book some helpful clues to understanding their children's progress and abilities and especially to understanding the important part which the apparent irrelevancies of out-of-school occupations may play in a background to mathematics.

I

FIRST IDEAS OF NUMBER

The unschooled learning of little children follows trends which have been the subject of much scientific inquiry, but the child himself when he enters the community of nursery or school, whether at 2 or 5 years of age, is highly individual both in his behaviour and in what he already knows. Children are, in fact, full of surprises; but that this is so is perhaps the least surprising thing about them, for not only are they individual in their powers but each develops in his own unique world of experience. Even within one family with its common home circumstances, each child has his own place amongst his fellows, and this personal environment is of vital influence in development. The teacher, like the gardener, is particularly concerned with the external conditions favouring growth, though the more he knows of the whole process the better. The period of germination of mathematical powers is not within the teacher's control, but when the seedlings are transplanted to his care it is the teacher's responsibility to see that they receive no sudden check and that any previous deficiency or forcing is dealt with appropriately.

Knowledge of the child's home background is an important guide to expectation. For instance, in some rough and· ready neighbourhood,

> Sally may go errands to the shops carrying a handful of pennies, no one of which she must lose. She may swing on the school rail while her older brothers and sisters are in school and enjoy the experience of an upside down view of the world. When they come home she may join in 'playing school' or be looked after by an attentive 5-year-old sister, and so begin to learn the speech pattern of counting.
>
> Pamela, however, may be the treasured only child of some flat-dweller. Mother takes her out shopping and tells her

> stories. If she is lucky she may share adult experiences in the kitchen of weighing and measuring and of putting dinner ready for three.

Her vocabulary may be larger and more reliable than Sally's, but all she learns is more guarded and selected and perhaps lacks some of the vigorous personal interest and independence of Sally's experiences.

Good nurseries combine the virtues of different ways of living, giving scope for the child's natural urges to learn and experiment and also providing the companionship and stimulation of a family with knowledgeable adults and children of different ages. Institutional life from babyhood may on the other hand show only too clearly how poverty of experience can limit development.

There can be no reliably useful expectation of the normal accomplishment of a new entrant to school. Handicaps due to poor environment may result in children being inarticulate and behind their potential best. On the other hand, children who have acquired adult forms of expression may give very misleading impressions of their capabilities. Counting, for instance, may, as described later, have very different meanings for different children, even in such limited contexts as counting fingers, toes, and stairs.

The word *two* is often used very early by children. That this should be so is not out of accord with expectation because pairs of objects are of very common occurrence; two eyes, two ears, two feet, two socks, &c.; two parts is the first result of partition (tearing in two); a group of two is easily recognized, and so on. But very little children also use the word *two* to serve associated purposes for which they have as yet no adequate vocabulary. These uses are interesting because of the evidence they supply of an underlying mathematical appreciation.

> Timothy, aged 2, said 'Two' when a second bird alighted on the sands, and then 'Two, two, two, . . .' as they came in great numbers. He showed a recognition of a two grouping and its relationship to many.

> Colin, rather older, extended to other even numbers the acceptable property which Timothy found in *two*. Colin could count in ones and could also begin to count in twos, 2, 4, 6, ..., &c. He refused all invitations to count 1, 3, 5, ..., &c., waving his hand aloft and saying 'One 's up there', 'Three 's up there'.

His feeling for these curious entities, numbers, marches with the ancients' view of the even numbers as 'earthly' and the odd numbers as 'heavenly'. Later he would catch his odd numbers and fit them into place among the familiar evens, which are associated with two feet on the ground, four legs to the chair, six handles on the drawers, and so on.

Although counting is a very early accomplishment, the absence or presence of a knowledge of it is not a safe guide to progress.

> Bobby, aged 4, did not know his number names but he laid the table in the nursery for eight friends with speed and precision. He placed a tray of mugs in the centre, stood and looked round, and finally remarked with satisfaction 'Them 's all for you'.

We cannot know how he assured himself that he had enough but not too many mugs. He may have matched a mug to each child, a fundamental number activity, or have matched them two by two as the children sat; but however he accomplished it, he tallied to his satisfaction a group of 8 mugs and a group of 8 children.

> Sonia, however, at the next table murmured to herself as she worked. She knew no English; perhaps she could count in her own language: but no, she was reciting rhythmically the names of the friends she served.

This formed a perfect system of counting for her purpose, but for her purpose only.

In contrast to Bobby and Sonia, who have rich experience of matching one group (children) with other equal groups (spoons, plates, mugs, &c.), two young sisters encountered in the train illustrate a different sequence in the approach to counting.

> The children's mother was talking with her husband and only occasionally with her children. The youngest, an infant in arms, followed in her happy crooning every intonation of her mother's attractive voice. The older sister, barely 4, while busying herself with pencil and chalks, also copied in speech almost every word her mother said. At one moment the mother turned to her children's entertainment with a nursery rhyme or two, ending with 'One, two, three, four, five, once I caught a fish alive!' The older child continued for a little while saying this to herself, finally breaking down into a jumble of other number names.

It was evident that she was well on the way to knowing the speech pattern of counting. It was a much-loved vocal exercise not necessarily associated with any other activity.

When counting is associated with an activity, such as going upstairs, the saying of *five* may signify 'I am on *a particular* stair (a wide one for example, that I know well)', 'I am on the *fifth* stair', or 'I have come up *five* stairs'.

> Brenda, who was 5 years old, was standing in a line of five children who were asked to count in turn; Brenda, being the fourth in the group, said 'Four'. The children were then asked to number from the other end of the line. Brenda refused to call 'Two', saying 'I'm four not two'.

The personal nature of the exercise may have focused her interest on the number as a label, or she may not yet have associated number with anything but labelling and have not had any sense of either order or quantity, i.e. of fourth or of the group of five.

Ordinary experience offers plenty of opportunity for making associations with counting varied both in type and in context.

> Tommy, a backward boy of 7, showed a serious lack of variety in his associations with counting which, however, he could pursue into hundreds. He was doing additions with three digits by making strokes and counting them and by using other meaningless tricks. He was, however, unable to count up the

> eight little ships stuck on a wall illustration of the seaside (Fig. 1).

To a practised eye the arrangement broke down easily into groups of four or easily counted rows. His counting was almost exclusively associated with counting strokes as he made them, and had very limited number significance.

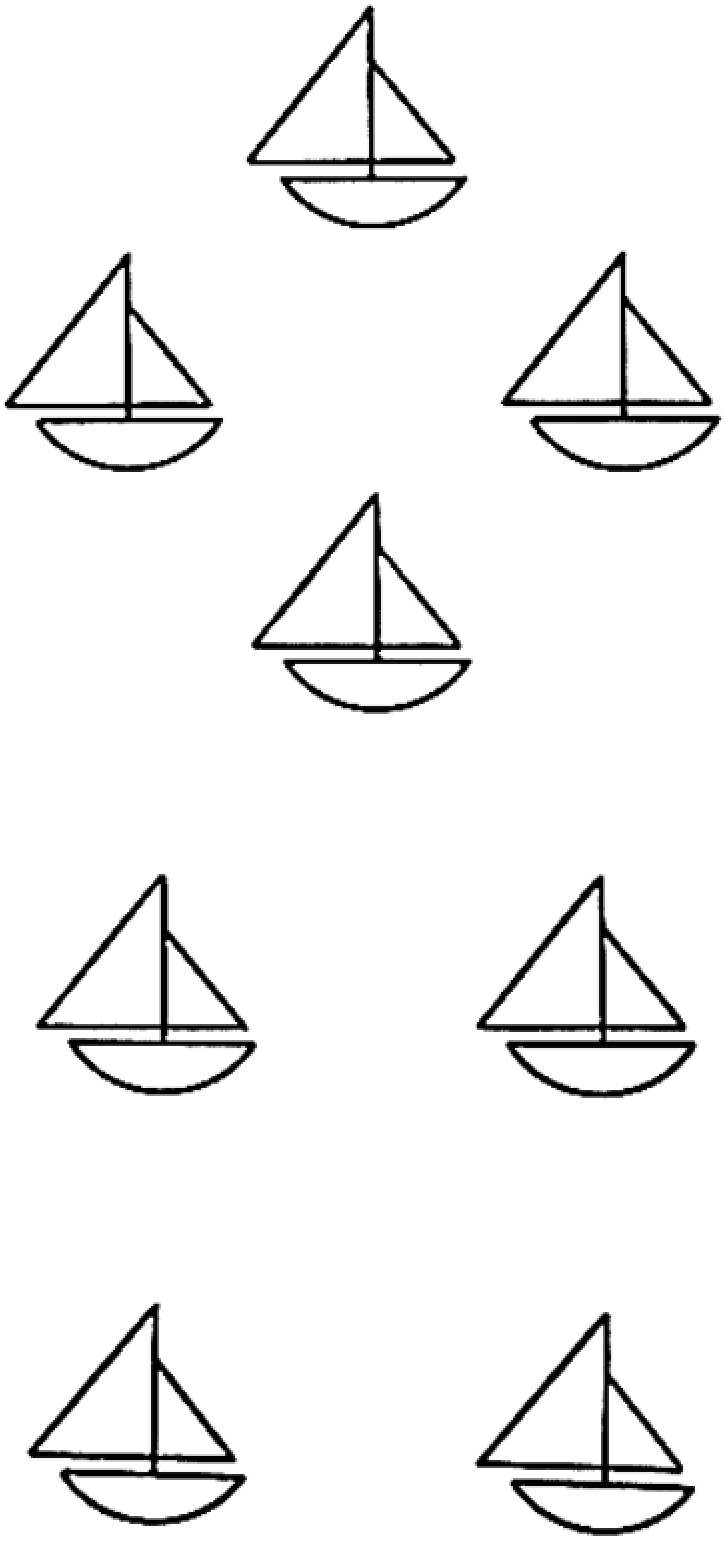


FIG. 1.

> Derek, on the other hand, aged 4, was unusually well advanced in the verbal ability to count and was making experiments in its significance. He had fourteen bricks; when he had counted them, he asked 'If I count these over and over again I shall get to a hundred, shan't I?'

Counting what can be touched is one thing, but counting what is seen but out of reach is another, and counting what can be heard only, like a striking clock or notes on the piano, is a still more advanced achievement which may have to be accompanied by nods or hand-clapping to make it feel like a personal experience that belongs to counting. These stages in the mere accomplishment of counting have possible differences of meaning, for example, counting claps may have the significance of either 'Make three claps; one, two, three', or 'Begin on the third clap; one! two! three!'

Quite young children who have reached the stage of using a number name for the number in a group may go on spontaneously as occasion arises to analyse, combine, and compare groups. Three examples of this are now given. All three children were under 5, and only one, Sheila, had been to school or had any teaching beyond learning to count.

> Arnold kept two goldfish which he fed on 'ants' eggs'. He explained to a visitor that he gave his fish six ants' eggs; three each, one for breakfast, one for dinner, and one for tea.

This knowledge of 6 as 3 and 3 in connexion with the feeding of goldfish, and of 3 as three ones, would be remembered by him, and since he was an able boy it might even at this early age have given him mastery over 6 in other contexts.

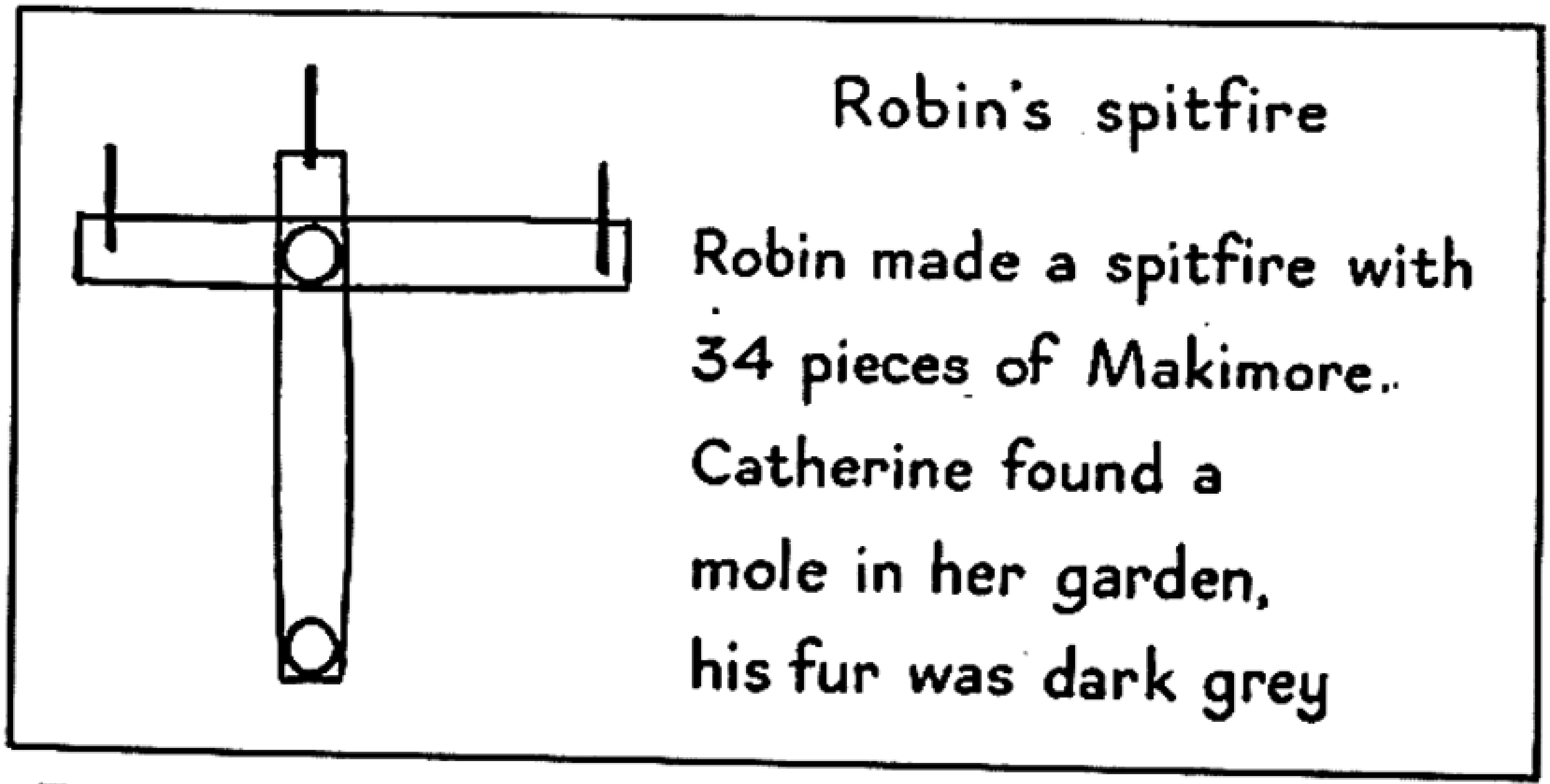
> Rosemary, who at 4 years old was already a bit of a naturalist, encouraged some older friends to collect the many-headed little yellow Lady's Slippers and to lay them out according to the number of flowers on a stalk as 'two-ers', 'three-ers', &c. A six-er was found but no one with more flowers until Rosemary turned up with a seven-er. On examination this was found to be a four-er and a three-er. Later a seven-er was found, but Rosemary's stratagem started a fashion for putting two heads together to form every kind up to a twelve-er.

Unfortunately when Rosemary went to school a month or two later she became confused with little sums such as $4+3 = 7$, for which general statement she was still unprepared.

> Sheila's older sister Celia had just had her seventh birthday, so Sheila remarked 'I'm 4 and Celia's 7, when I'm 5 Celia's 8, and when I'm 6 Celia's 9. She is 3 years older than I am'.

Sheila's comment shows a fine sense of the constant interval between the ages, and it is probable that she found the 6 to 9 interval easier to cope with than 4 to 7 or 5 to 8; but even allowing for children's great interest in ages and for the effects of teaching, possibly of counting in threes, a step of three at this age is a great achievement, especially in such an imaginative context.

A number of the stories told are of able young children because they exhibit the kind of progress that can be made under the stimulus of ordinary living conditions, when the mind is ripe, with very little teaching beyond the accomplishment of counting. This progress is not achieved without adult interest and sympathy. Any age-group of children will show a wide range of background and of maturity, but any rich environment will provide endless opportunities for exercising and enlarging number knowledge.

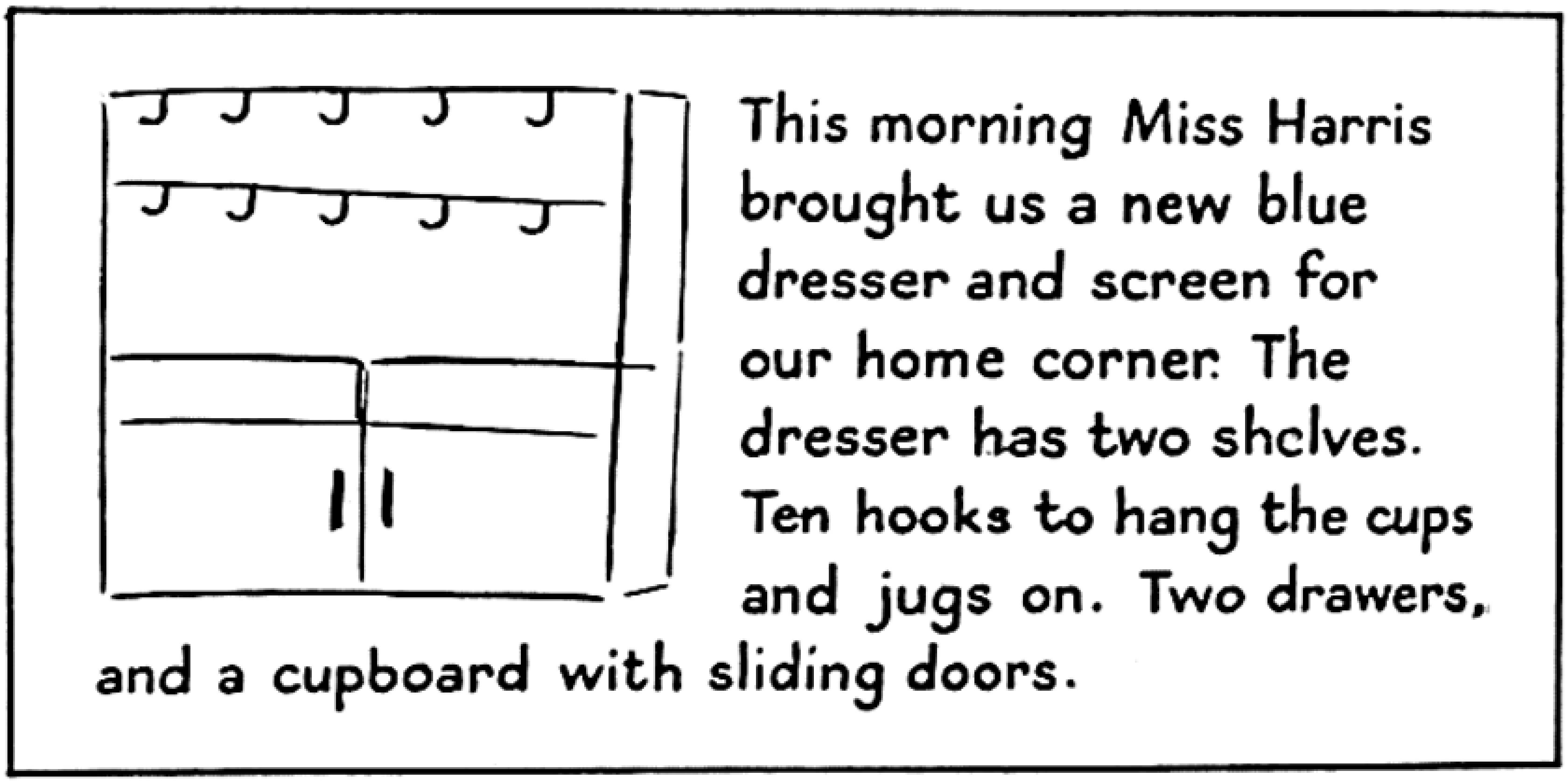


Robin's spitfire

Robin made a spitfire with
34 pieces of Makimore.
Catherine found a
mole in her garden,
his fur was dark grey

FIG. 2. Pages from News Book (composed by children but written by teacher).

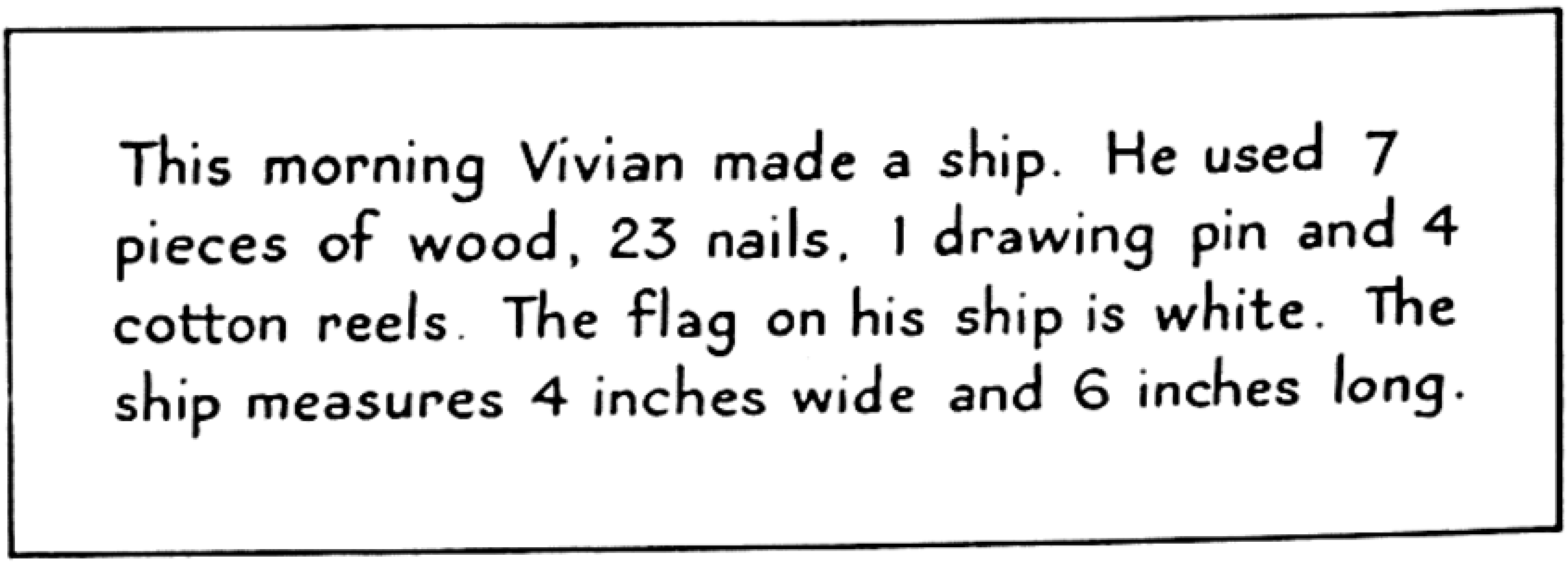
Figs. 2, 3, and 4, show pages from news and record books belonging to a class which had only been a week or two in school. The children were well developed for their age, and a fashion for counting and accurate description showed itself more often than not in the items of news recorded. Further examples involving measures are shown in Fig. 13 (Chapter III). Although interest was in the first place stimulated by the teacher, later responses were spontaneous. Experiences were shared with the children without guidance about what was to be inferred from them. The children naturally did not all react alike. The contributions by Robin of a Spitfire of many parts and by Catherine of a mole with



This morning Miss Harris brought us a new blue dresser and screen for our home corner. The dresser has two shelves. Ten hooks to hang the cups and jugs on. Two drawers, and a cupboard with sliding doors.

FIG. 3.

grey fur show divergent interests, though each would hear the other's contribution.



This morning Vivian made a ship. He used 7 pieces of wood, 23 nails. 1 drawing pin and 4 cotton reels. The flag on his ship is white. The ship measures 4 inches wide and 6 inches long.

FIG. 4. Page from 'A Record of things we have made'.

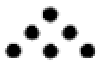
If natural opportunities for counting and matching groups, cups with saucers and so on, are in sufficient variety, many children will not need special counting apparatus. Counters or some other tallying apparatus should, however, be a part of the equipment of the classroom readily available for a child's use if he engages on some number adventure which is beyond him. Apparatus may also be used to help the child who is having difficulty in reaching the stage of detaching number from the particular situations in which numbering occurs. When numbering things, the placing of counters to stand for first, these objects and then, in different circumstances, for those, helps to give generality to numbering and to raise, for example, the number five to the status of something which can be thought of as characteristic of any group of five objects. Prolonged use of special apparatus is, however, generally to be avoided; difficulties may be due to an unreadiness better cured by time and maturation.

The device of arranging spots or objects to form a pattern is often used to emphasize the key stage of associating one group with an equal one and with the appropriate number name. These patterns also bring out the relation between one number and the next; for instance, the domino five ⁙ includes the domino four ∷ and one more spot.

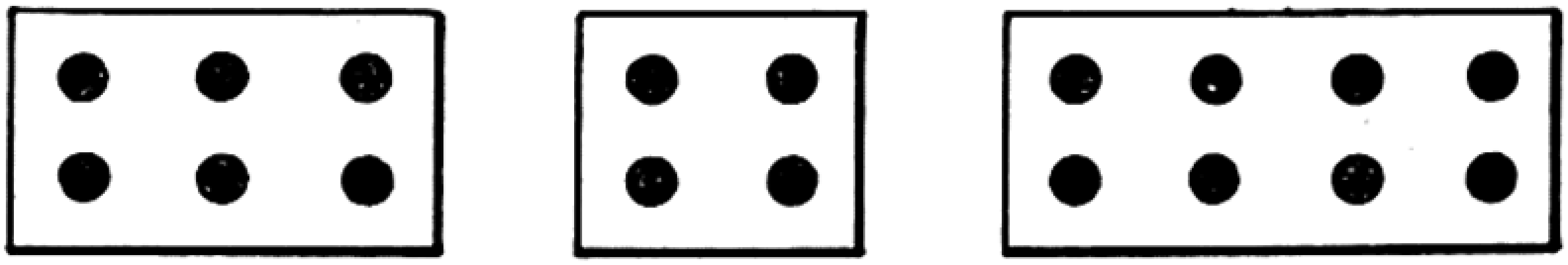
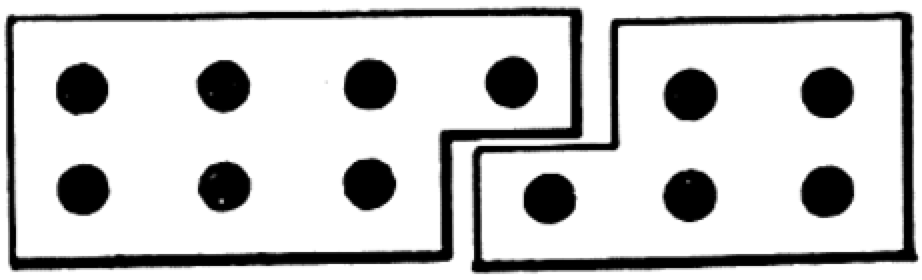
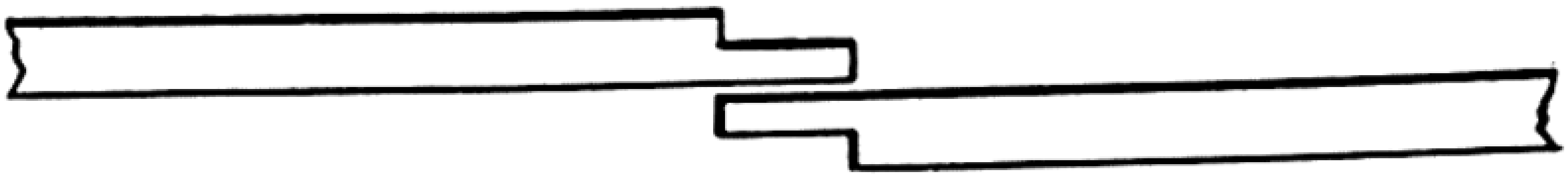
Although patterns can play an important part in learning about numbers, there are dangers of over-use, especially if one type of pattern is allowed to become dominant.

> John, a very backward boy of 10, exhibited the domination of a pattern to such a high degree that it seemed as though he had a spotted mind. He could neither count nor work any practical problem without moving counters into patterned groups.

The pattern had indeed become a reckoning device which had far outlived the purpose of exhibiting number groups. He did, however, show some command over the stage of simplifying a problem by making a counter take the place of something in the problem which was distracting to think about. Abler children shed any such obvious evidence of the persistence of the pattern as an intermediary device. However, when so-called number pictures are used till they become a habit, traces of their interference in thought and reckoning at stages when they might well be discarded can often be seen in head, hand, or eye movements.

Different patterns have their own different characteristics, so that even if one type of pattern is used more than another for group recognition it is wise to allow variety and experiment. Children who build up their own patterns from particular numbers of counters may arrive at results which are unlike any which they might encounter in, say, using dominoes or playing cards; for example, this pattern for 6: .

One type of pattern may have a particular fascination for a particular child.

> Fanny, aged 3, was attracted by the cards which her older sister was using. These were made on the two-by-two patterns in which each even number is shown on a rectangular card (Fig. 5), and each odd number on a stepped card (Fig. 6).
>
> 
>
> FIG. 5.
>
> She knew only one number name, 'Portment', her name for the ten card which she had heard described as 'Important', and she knew she had portment fingers and portment toes. Her special interest was in fitting two odd number cards together to make an even number (Fig. 6).
>
> 
>
> FIG. 6.
>
> This conformed to her general interest in fitting things together properly, observed before she was two when she was unsatisfied with a garden gate closed wrongly (Fig. 7).
>
> 
>
> FIG. 7.

When she built up an even number she related it to the Portment card and demonstrated, for example, that fourteen was four more than ten (Fig. 8).

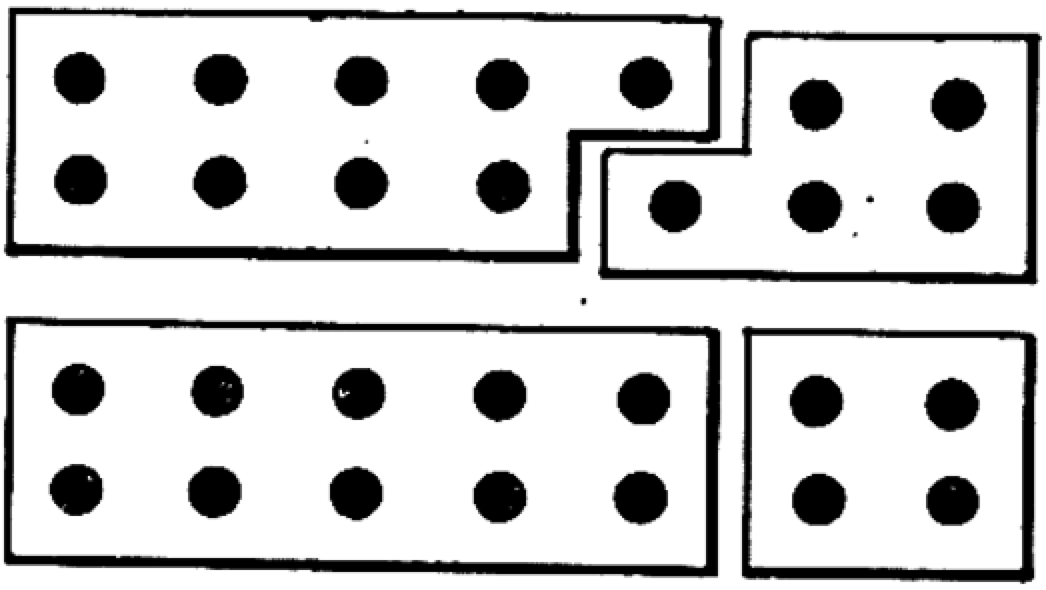


FIG. 8.

Who shall say what this child knew about numbers before she could count?

Systematic types of apparatus, such as number-picture cards, depend on wider experience to give them life. They stand between that experience and the number names or the written figures which are the subject of the next chapter. Counters, shells, sticks are all the servants of experience; they are useful means of recording a count. Arranged in a pattern they do exhibit something more than a mere count. Patterns exhibit something of the properties of particular numbers, whereas an array of counters just helps to isolate the number property from a more elaborate situation. The danger arises when devices which help to clarify a situation become, not a means of experiment, but a habit associated with reckoning. Although reckoning other than counting belongs to a much later stage, its forms and ritual often obtrude themselves into classroom practice before the idea of a numbered group has had time to take root.

Materials in a classroom may be definitely suggestive of number without being in any way formally directive. Sets of dominoes take a high place amongst these, though they are as likely to be used in the first place as building bricks as for their intended

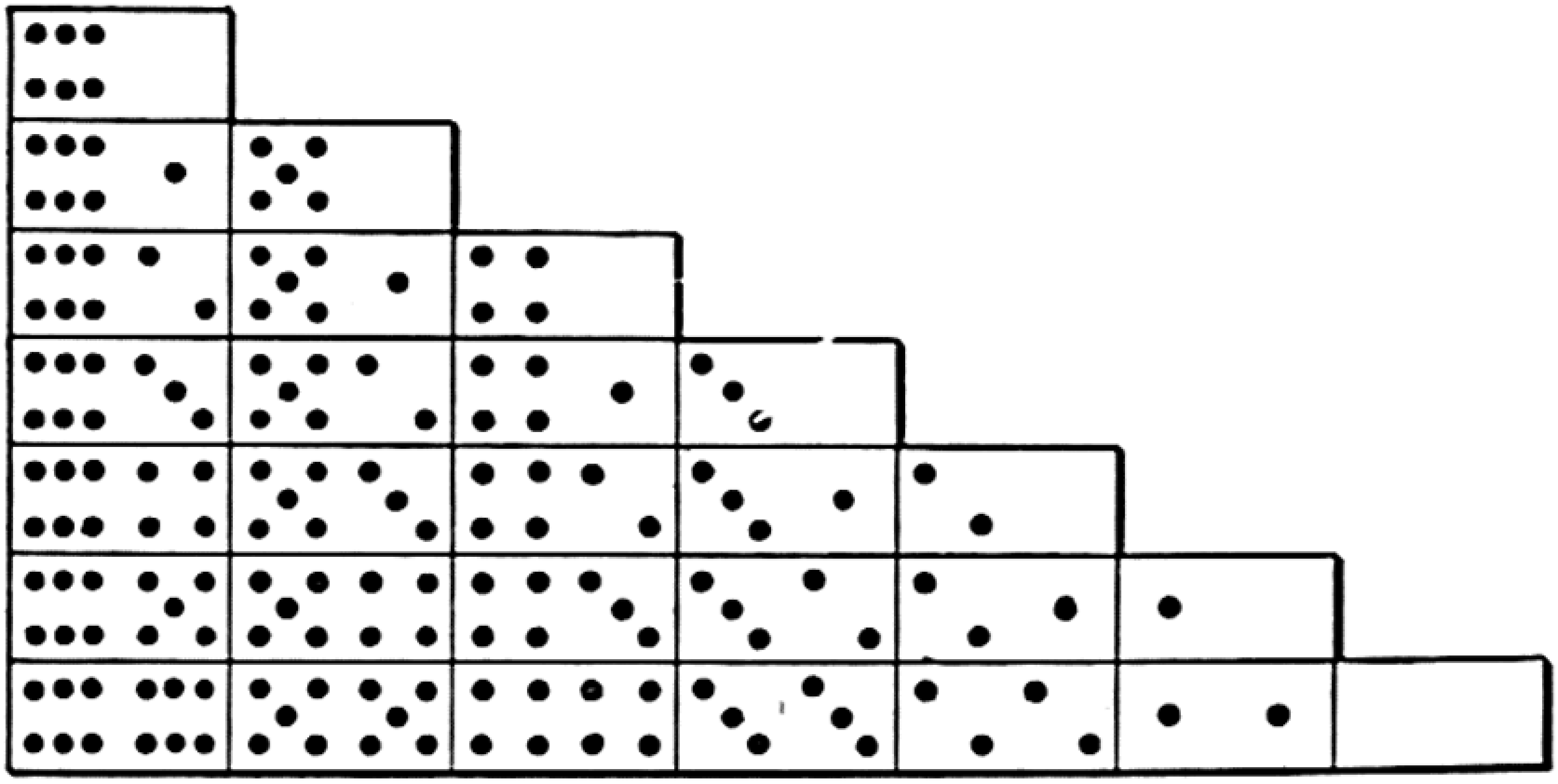


FIG. 9. Domino stair, No. 1.

purpose. Matching pattern to pattern has an attraction at an early stage not as a game but merely as an experience. The occasional naming of the pattern, 'Another six to match this one', by the teacher as she passes increases awareness and precision.

Fig. 9 shows an arrangement of a full set of dominoes which is attractive to quite young children as well as to older ones.

> Two small boys, one of just 6 and one a little younger, who were accustomed to the look of dominoes, readily hunted those with the pattern ⠿ on them, but in a first arrangement of the six column did not get the sequence as shown. An older child later put this in order. The young pair were able to complete the stair in some fashion using the remaining fives, fours, and so on in turn. They called it a stair, and one of them picked out the middle-sized stair spontaneously, counting 1, 2, 3 from each end as a check. Some older children, looking on, not only put the stair tidy by arranging each column in sequence but took an interest in turning the steps into a rectangular block by levelling up to the middle stair, as shown in Fig. 10. They also played the game 'Is the set complete?' and 'Which domino is missing' by trying to complete Fig. 9.

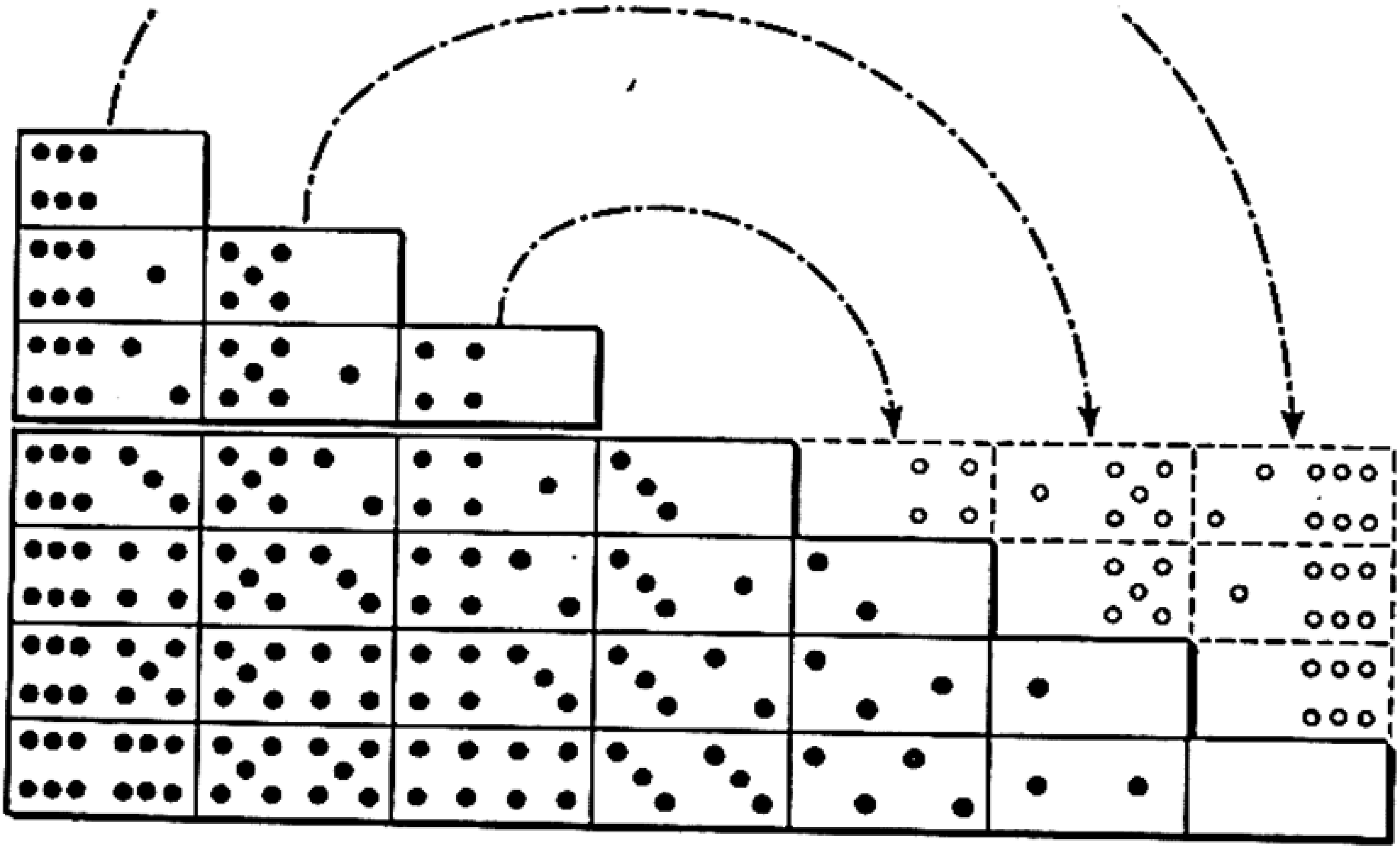


FIG. 10. Domino rectangle.

Another domino pattern is shown in Fig. 26, Chapter V.

> Sarah, a systematically minded girl of nearly 8 who was familiar with dominoes, when asked if a set was complete spontaneously made the arrangement shown in Fig. 9.

She not only had a good command of pattern and sequence, but she realized that a count of dominoes was not enough.

A final illustration may serve to emphasize this important stage of identifying a group and naming it.

> Stella, aged 4, liked to identify playing-cards for her father but could not recognize the nine. The game was abandoned for a month after which, at a fresh trial, nine was recognized without hesitation and with no intervening practice specially directed towards this number or indeed to any type of recognition game.

II

HOW NUMBERS ARE WRITTEN

The previous chapter has dealt almost entirely with the early verbal approach to number and with even earlier pre-verbal stages. Almost any environment, however, except that of the child in a meagre home in a rural area, will confront children with written numbers, often in circumstances of interest to them. Figures occur on house doors, shop prices, omnibuses, clocks, watches, calendars, or the pages of a book. If a child is in company with a sympathetic adult or an older child he may learn that his house is number nine or the bus passing the end of the road number twenty-three. He will learn to associate the figure shapes with these number names, and the written symbol will begin to share in the meaning attached to the spoken word. Opportunities may become more frequent in the schoolroom. The news book, pages of which are shown in Figs. 2 and 3, is a good example of extending this recognition of figures to a record made by the teacher of the children's own observations. Many examples can be given of familiarity gained under conditions of natural experiment in a suitable environment.

> A class of children of 5 and 6 years of age of low intelligence and poor background made free use of a wooden clock face, which was in great demand. The figures could be taken out and replaced. The children frequently occupied themselves doing this. Mistakes in replacement were made more often than not, but corrections were made only by other children in the group, not by the teacher.

Gradually many children acquired familiarity with figures and with arranging these written symbols in order up to twelve. This was not their only association with written numbers, but formal apparatus was not used in this class.

> Alfred, an abler boy in the class, carried his interest in the clock much further, acquiring considerable skill in dealing with minutes past and minutes to the hour.

> Beatrice, a child in a 5-year-old class, used some old rulers as building material. She took a fancy for drawing rulers and copied the ruler, its main markings, and the figures on it, with considerable skill. This started a fashion for drawing rulers.

Again, the children of this class learned figure shapes as an incidental activity.

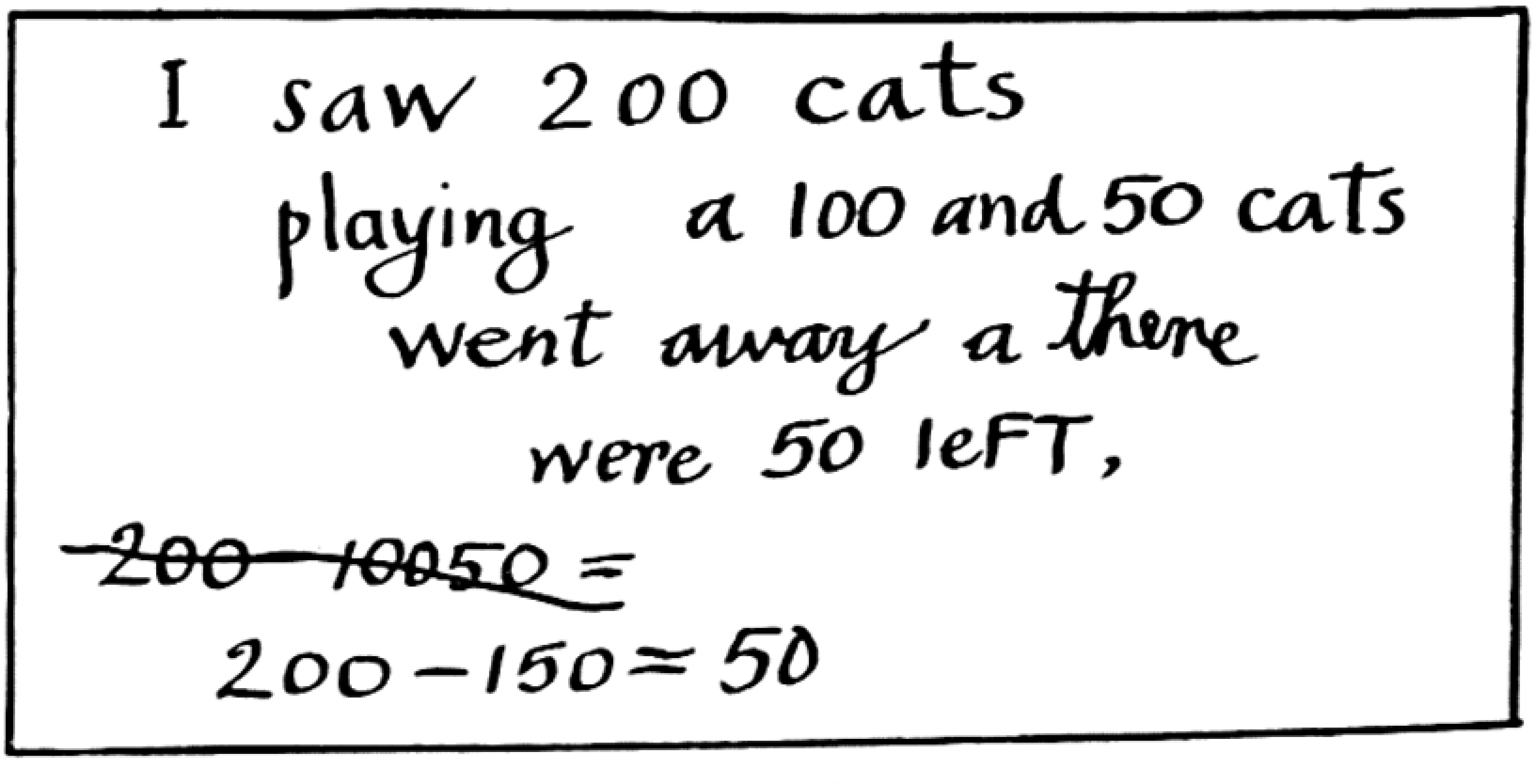
Nevertheless, the reading and writing of figures is a stage which marks a level of accomplishment. Figures will in time mean the same for the children and acquire the same associations as the words for which they stand. Much of the apparatus in use in schools for teaching quite young children is directed to the teaching of figures and only incidentally to number experience. It is a great convenience when figures are known, but at the fundamental stage when experience and meaning are the first concern, the spoken word is symbol enough. The written figure symbol would be altogether premature at this early stage were it not already a familiar part of an ordinary environment. The elaboration of apparatus at this stage is generally a sign of trying to do early what could be done without fuss if it were left a little later.

The reading of numbers over nine begins quite early, perhaps concurrently with the recognition of figures. The known numbers of house doors, buses, and other familiar things form a nucleus round which other less familiar shapes can be fitted and named.

> A girl of pre-school age was overheard talking to her mother on a bus which was temporarily marooned for a change of driver. Her mother entertained the child by commenting on other buses which passed them at their stop, referring to each by its route number. Presently the little girl asked 'Mummy, has our bus got a number ?'

Few of the so-called large numbers occurring in this early observation have direct quantitative associations. Book pages are ordered, but the bus numbers are mere labels giving a number-naming of the bus. Even the calendar names the day rather than marks the passage of the month. Apart from the 'teens', with their reversal of the tens and units order, the counting names are a help once a beginning has been made. The point to be noticed is that the written symbols may be put into words and invested with any associations which those words hold with no appreciation of a grouping in tens other than the rhythmic tens grouping of counting itself. The written number 23 is read as a whole twenty-three, not as twenty and three. There is a temptation to assume that naming a written number means more than it does; the naming of larger numbers is common amongst infants, and it is not generally realized at later stages how little this may mean.

> Michael, aged 8 and fairly able, was inventing problems for himself. He decided that there should be 200 cats in his garden and that 150 should go away, leaving 50 behind. His efforts to set this down are shown in Fig. 11.



I saw 200 cats
playing a 100 and 50 cats
went away a there
were 50 leFT,
~~200 10050~~ =
200 − 150 = 50

FIG. 11.

This gives a pretty clear indication that his adjustment to tens notation was as yet incomplete. The guiding influence of the 200 is interesting. He felt the incompatibility of 200 with the five digit 10050 which should be less than 200, so he crossed through the line and rewrote it, not only leaving out the *and* in 100 *and* 50 but also slipping his 50 into place. The self-set nature of the exercise led to the discovery of both the trouble and the remedy.

There are then two points at issue; one is the extension of experience to cover some understanding of 'larger numbers' and their uses; the other is the increase of awareness about how we write numbers, from the point where the child starts with an almost complete acceptance of written numbers such as 23 as items of experience. Derek, in Chapter I, had achieved some idea of 100 as a number reached by the repetition of counting, his scale of 14 being purely incidental.

> Alec and Simon, aged 6, counted the words in their handwritten 'dictionaries' and reached two hundred and thirty-seven.

This passion for counting, had it been applied to groups of ten words, might have led to a more revealing activity. The exercise was entirely self-set.

> Nigel, not yet 5, had a glimpse of infinite series when he remarked that his counting could go on and on and on.

> Bill, aged 6, had a formal job which absorbed his attention; there was a clothes basket full of cotton reels and ten upright sticks, each of which held ten reels. Every time he put on a reel he wrote down a number, keeping to a separate column for each stick. He had reached 78, and when he was asked about what he was doing he remarked, 'When I've put on two more I shall have 80, and I'm going on till I get a hundred'.

The note of satisfaction in his voice has to be heard to be appreciated. It was a stout job, and he felt his gain in mastery as he did it. The exercise was not premature because he had a genuine interest in written numbers.

In contrast to Bill with his formal exercise, two children at a bus stop offered an almost perfect example of a spontaneous learning situation.

> A boy aged perhaps 8 and a girl of 6½ years were waiting for the bus and playing around as children will. Presently the boy produced his bus money and remarked 'I've a Victoria penny'. The girl looked at hers and said. 'So've I'. 'Mine's eighteen-sixty-three', said the boy. 'Mine's eighteen-ninety-eight', said the girl. Then followed a swift argument which died down as quickly as it flared up. 'Mine's older than yours', said the boy, 'No mine is! 'No, it isn't.' 'Yes it . . . Oh! I see', capitulated the girl.

There was no doubt that she did see and that she had made the difficult adjustment that the larger number 1898 on the coin meant a newer coin. She probably disregarded the 18 in both numbers, and it would be interesting to know whether she could have negotiated the turn of a century. The labelling of coins (or of the years) is a difficult and extended use of number, for it implies an origin or zero, from before and after which years are labelled, an idea which is not normally dealt with explicitly in primary schools. This idea of an origin is different from the nought which appears as a place-holder to maintain the position of other digits in written numbers, or from the nothing which, for example, occurs in the Nigger Boys 'and then there were none'. Moreover, anything to do with time has its own difficulties, and the idea of an older or newer coin implies a change of view point from the origin of numbering to the present. There is, too, the difficulty of labelling a whole year with the same number. The adjustment made by the child of six was of course an intuitive one made with the aid of her ability to count and to read numbers and her experience of pennies, but it was a notable achievement.

Acquaintance with larger numbers of which some particular examples have been given is more often an acquaintance with labels, like the numbers on an engine or a penny, than with quantities. This makes the task of keeping large written numbers in touch with meaning a hard one. The foundation of understanding of the sophisticated idea we call 'place value' is, however, being laid in simple form before the child directs his attention to the special way we write a group of ten. Concurrently with the child's uncritical acceptance of, say, 53 as the way to write fifty-three for the fifty-three bus, he is using small numbers with a group significance. Threepence is neither a label nor a third penny but a group of three coins. The next stage, that of accepting a threepenny bit as an equivalent of those three pennies, is still another step towards using a symbol to stand for a group. Taking two threepenny bits stands for two such groups, and so on. Similarly, X in Roman numerals stands for ten and XX for two tens. The size of the group is immaterial. Derek was thinking of a repeated group of 14 when he counted to 100, but it might equally well be any number, 2, 5, 10, 12, or what you will. The exercise of counting in groups and recording the number of groups counted is, in fact, an exercise in scales of notation. It is dealt with later as the multiplication table. There is, however, this quite early stage appropriate to young children, namely the acceptance of a single token, a counter for example, to stand for a group instead of for one object. A two-shilling piece is such a token for two separate shillings.

> David, quite an able boy of 6, who could count in twos quite readily when counting separate shillings in pairs, had a difficult adjustment to make before he could use the same series of numbers to apply to the number of shillings when the coins were two-shilling pieces, although he could quite readily exchange large coins for their equivalents of smaller value.

Ten-shilling notes are with devaluation less remote from children's experience than formerly. They offer a useful example of a token for ten, though there is always available two clasped hands as a symbol for ten fingers.

Some form of apparatus, bundles of sticks, or bead bars, or an abacus, which actually contains the group, are easier than the single counter of different size or colour to stand for a group instead of the unit. Both forms of apparatus may well have usefulness over a period of years while appreciation grows of the way we make the position of the figure indicate the group. Apparatus used experimentally as interest and need dictate is better than an attempt to deal at all exhaustively with notation at some particular stage.

A good example of the use of environment instead of special apparatus occurred in a school with a terrace of square paving stones outside the classroom windows. In one place this terrace was ten stones wide. It was therefore possible to count a class or more of children by sending them out to fill up rows of ten paving stones, so exhibiting their number as so many rows of ten and so many extra.

The number of children in a class or two, is as big a number as young children are likely to deal with in any quantitative way. But, as Nigel showed with his counting on and on, children reach out at least in counting to something which they feel is very, very large and beyond them. Large group names such as thousands and millions come to their lips with strong feeling but no exact appreciation, though alternative naming of large numbers may have some meaning.

> When asked about large numbers, an able class of young junior children claimed familiarity with a thousand. It was decided to count to a thousand in tens. At 200 the counting was stopped and the group changed from tens to two hundreds. The children chanted happily two hundred, four hundred, six hundred, eight hundred, ten hundred, and were just passing on to twelve hundred when a smart boy thrust his finger out as a pointer and cried 'A thousand!'

This boy's appreciation was of a different nature from that of the children who ranged themselves on the paving stones. It was hardly quantitative at all. It was a moment of insight into the pattern and power of this number notation, shared in some measure by the class as a whole, but memorable for the lad who saw it for himself.

The same kind of enlightenment has to come at the hundreds level also.

> Bertie, about a year younger than the children described above, had some acquaintance with written numbers of three digits. He and his neighbour were using peg boards containing ten rows of ten holes. He counted the holes in his board in tens to one hundred, and then set out to count his neighbour's board also, '. . . ninety, a hundred, a hundred and ten, . . . a hundred and eighty, a hundred and ninety, *a hundred and a hundred*'.

In this difficult matter of how we write numbers a plea must be made for the child of less than average ability. When he knows his figures to 9 there is a risk of assuming that he also has a fairly full appreciation of what they stand for, whereas they may be mere equivalents for words which follow in succession and still lack meaning as numbering a group. Again, when two- or even three-figure numbers can be given their proper verbal equivalents, it does not follow that there is any appreciation of their composite nature, which involves an act of addition (not yet discussed). Were it not for the fact that the first acceptance of such a number as 23 is a wholesale one, the discussion of two-figure numbers would be out of place in an early chapter. The acceptance of nought as part of the way we write 10, 20, &c., also needs no analysis at an early age. The difficulties which occur later in, for example, writing numbers as shown in Fig. 11 with the 100 and 50 cats are first overcome by a simple awareness of how the conventions for writing numbers work. Children accept written numbers as part of environment, and then get into endless difficulties because acceptance is mistaken for a much more mature understanding.

III

MONEY AND THE BEGINNINGS OF MEASURING

> I have seen in print disparaging comments on play methods of introducing children to arithmetical ideas. Such criticisms seem to be founded on an erroneous assumption that the play methods are intended to teach something about arithmetical procedure, for which purpose they are, as the critics perceive, ludicrously inadequate. But these methods have no relation to the teaching of anything; their use is to give to the infant brain a start on a line of development such that, in after years, when intellectual teaching begins, it will be capable of receiving that teaching.
>
> MARY EVEREST BOOLE
>
> *Preparation of the Child for Science, 1904*

A FEW examples in previous chapters have been concerned with money or measures, but it has been for the number element in them that these have been quoted. It is now proposed to consider the early approach to money situations a little more systematically, and to discuss how children adjust themselves to the ideas implicit in the use of measures by approaching the question from the point of view of what children round about nursery and infant school age do and observe.

Money

An interest in shopping may begin very young, and shopping is so important that activities simulating shopping of one kind and another are often continued late into primary school life. Most children except, maybe, those in the depths of the country, know something of coins and their uses before they reach school age. Where an interest in pennies already exists, buying objects at a penny each may well provide one of the first approaches to number. Tallying objects with counters is a very similar operation to buying penny objects with pennies. Buying at twopence each, however, immediately introduces a group counting, the mastery of which cannot be hurried. Some young children, in a poor neighbourhood where children go shopping errands from an early age, learned much from buying penny and twopenny loaves, batches of which were made as class apparatus with flour and salt by a father who was a baker. The twopenny loaves of Fig. 12, which is a spontaneous recording in a 'free' moment, followed after buying penny loaves, which were made in one piece instead of the two pieces of the little cottage loaves costing twopence.

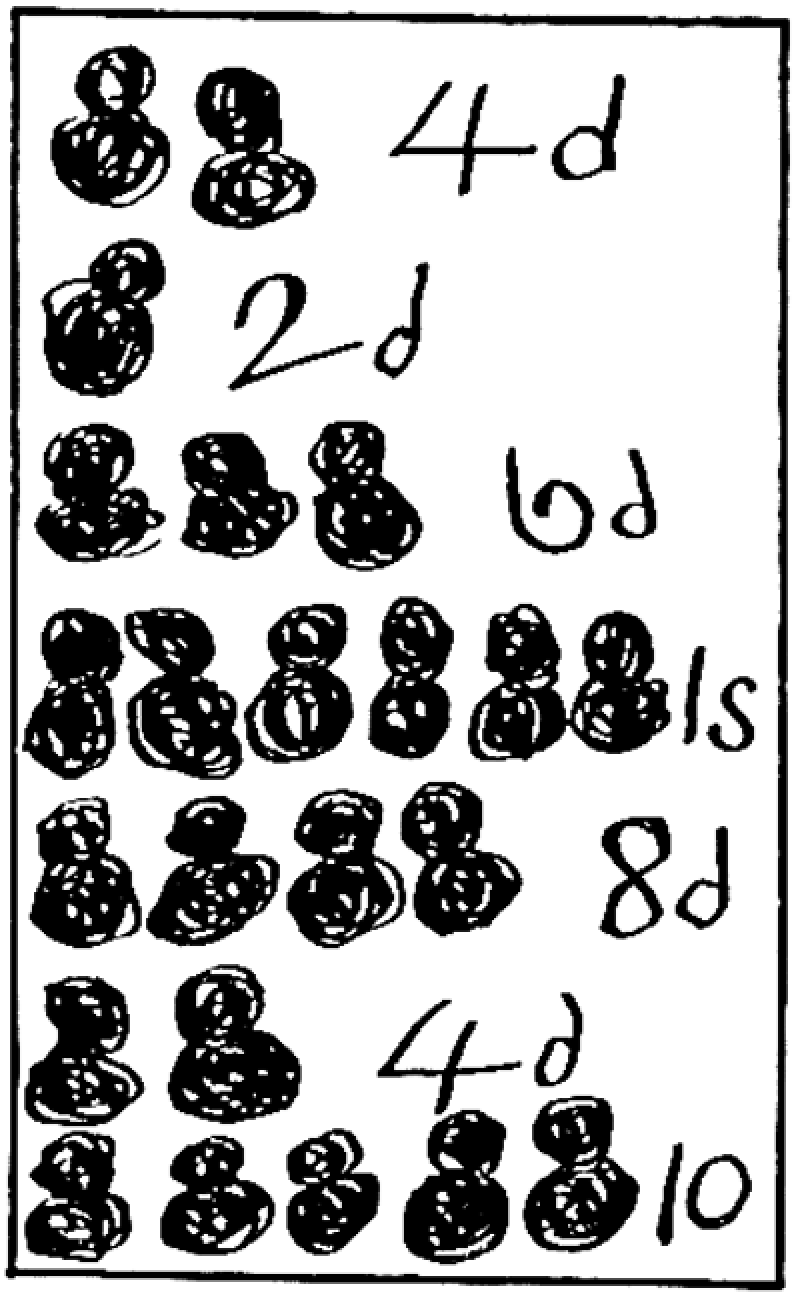


FIG. 12.

The use of threepenny bits, as has been shown in Chapter II, introduces a group token, which is a further step in understanding.

Familiarity with coins themselves is an important part of children's general knowledge. This cannot be learned properly from cardboard coins, and it is possible to organize the use of real coins by at least a few children at a time without undue risk of loss.

The children whose work is shown in Fig. 12 above used small boxes containing twelve real pennies each. They were so distressed when one penny was lost that a quite poor boy brought a replacement in the afternoon. Fortunately cleaners found the lost coin.

The old game of scribbling an impression of a coin is a good way of familiarizing rather older infants with the details and relative sizes of coins. If the coin is placed in the corner of an envelope, the skill needed to produce a good 'rubbing' is not very great. A row of rubbings of different coins can be obtained along the bottom or side of the envelope by folding to make new vantage points for holding the coins.

The placing of groups of coins equivalent in value to one of higher denomination is an important experience. The groups of coins may be of mixed values or in piles of equal coins. Shopping experience supplies motive for acquiring a ready knowledge of equivalent coins. The counting up of money follows on from this. The use of two fingers for counting similar coins in pairs, like a cashier, is an attractive way of encouraging the pairing of coins for a double count. It may or may not introduce difficulties: counting pairs of shillings is easy, but in counting sixpences or two-shilling pieces counting up the number of coins is different from counting up the value in pence and shillings. The equivalence between, for example, 6 pennies and 1 sixpence is arbitrary, but it is acceptable as a matter of experience.

Counting and shopping with money up to a shilling gives extremely valuable experience not only of money itself but of particular number experiences within the grasp of many quite young children.

Imitation of shopping activities may also introduce acts of weighing and measuring. These will be referred to again later in the chapter. The kind of shopping which is of greatest interest to the children varies with age and circumstances. The sweet shop and the toy shop attract at an early age. The grocer is generally interesting at an earlier age than the draper, who usually comes into his own at 7 or 8 years. Milk bars, teashops, and markets have their day. The tea shop, in particular, pleases because it offers both waitress and customers opportunity for dressing up, even though lack of skill with numbers may decree the cheapest of meals.

Measures

Experiences, common in nursery schools, which have a bearing on ideas of shape, size, and quantity may have been comparatively scanty in the home experience of many 5-year-old entrants. It is important that this lack should be made good. The child who has opportunities out of school of sharing with parents and friends rich experiences of the home, the field, and the workshop has an incalculable advantage. It is difficult for the adult to realize how much of his own past he brings to the interpretation of the present, and therefore how much an infant depends on the range of his opportunities and, indeed, to how late a school age actual physical experiences determine mathematical receptivity.

> Malcolm, scarcely 3, who was unusually articulate for his age, looked out of the sunshine roof of the stationary car and saw clouds scudding behind the roof of a house. 'Why', he asked, 'does the house look like falling on the car?'

He was making an adjustment which the adult mind makes without effort, except in such cases as the relative motion of trains when either or both may be moving. In nursery or reception classes experience of water, sand, dough, wood, paper, and other materials begins as experience *of* materials themselves, but soon the emphasis changes to experiments *with* materials, and provided there is access to a variety of what in a wide sense may be called tools, this phase may remain incomplete for a long time. Tools here include household and natural objects, toys, &c., as well as the means of construction. Experiments with liquids, for example, may occur either in the course of imaginative play in the house corner with various cups, teapots, saucepans, &c., or in the more purely experimental setting of water-play in which containers of various shapes and sizes may or may not have a message for the experimenter according to his stage of development. The child who submerges a medicine bottle and who in trying to fill it reaches an effective position and cries 'Bubble! Bubble! Bubble!'

is having a prescientific experience which he will repeat to his heart's content. He may on the other hand inquire about the dosage marks on the bottle and so start a train of interest in 'How much ?'. But when he pours out liquid from the bottle it looks a quite different amount in its new container, and at early ages this is a great stumbling block. This idea of sameness of capacity with difference of shape is, indeed, one of those difficult ideas which adults have accepted so long that they take it for granted. Children, however, must learn it from experience.

> Christine, when she was aged 4, was much disturbed because her older brother Robin had a tall cup while hers was wide and low. She insisted that he was having more to drink than she was. Actually the cups held the same, but no demonstration of this satisfied her.
>
> When she was turning 7 she went with her mother to see a local pottery and was given the choice between two mugs differing in shape, like the cups at home. She hesitated over the choice, then, remarking with complete acceptance 'They will hold the same', she chose the taller mug.

Perhaps this was a purely aesthetic choice, the tall mug was primrose yellow inside, or there may have been lingering preference for the long drink, seemingly the larger. Her mother's comment that 'when Christine was 6 you could tell her things like this' (that is, that mugs held the same) is a useful pointer.

> A little later Christine made use of a medicine bottle marked in tablespoons and filled with coloured water to treat her sick doll. She remarked 'Are these really tablespoons ? They look so small'. But she was convinced about the amount by using a tablespoon to administer the dose. She also took the greatest care not to spill any. She poured the dose away after the mock-dosing, remarking 'I must not spill any on Pamela's woollies as it is bad weather for washing them!'

Experiments with dough and modelling materials again give experience of constancy of substance accompanied by change of shape, but since the form is more in the control of the experimenter the awareness is more likely to be of shape than of substance. The lump rolled out into a long snake or flattened into a pie-crust is an experience the fascination of which even an adult can remember. The cutting of round tarts from the sheet of dough, with the inevitable wastage and re-rolling, has implicit in it an experience of the circle as a satisfying form but an unsuitable unit of area. The act of rolling a lump into the semblance of a ball needs a muscular control which gives a feeling for its perfect symmetry; undue pressure on any part of the surface deforms it. To stick in a thumb and turn a convexity into a concavity, all smooth and continuous with the rest of the surface, gives the experimenter an experience of an inside which is still outside, unlike the plum inside the pudding. Indeed modelling at all levels can be a geometrical as well as an artistic experience. Play with sand also offers many experiences akin to those with water or dough. Dry sand can be used easily to fill and even to pour. It is sometimes used, for example, in a teacup where its colour helps make-belief in spite of its dryness. Wet sand is more retentive of a shape and so has more in common with some experiences of modelling.

Wood, on the other hand, is a more intractable medium, and any change of shape or size needs a more deliberate approach. But wood can be treated with considerable violence without breaking up, and probably for this reason, and because of imaginative imitation of male adults, boys usually like using wood, either for doing simple woodwork or in the form of bricks, boxes, and other building material. It is not easy to say whether girls on the whole take to all this a little later or whether the boys tend to crowd them out from the use of this attractive material at the earlier stages. The importance of bricks and rigid constructive materials for pre-mathematical experiment is so great that the opportunities and interest trends of both sexes should be watched carefully. Often, for instance, girls might use building material for home corners if it was available for them. They undoubtedly like woodwork if opportunity is made for them, but the imitative urge being probably a little less strong for them than for the boys, they would be likely to be attracted to woodwork rather later.

Early experiments with building bricks may, like a child's early drawings, only bear chance resemblance to particular objects or structures; forms are put together and then perhaps named because of this chance resemblance. The deliberate association of forms to represent, for example, a building shows a much more advanced stage involving both a breaking down and a building up of a complex shape; sometimes this is done by treating bricks just as bricks and sometimes by taking single pieces of building material of appropriate shape to represent whole walls or roofs.

> Rosemary, aged 6, was given a box of bricks which had a book of model houses with them to be copied. The copying involved a careful selection of sizes for which there was a key, which she very seldom used, and some of the bricks had to be carefully orientated. She showed good judgement of distances between walls and good powers of dealing with bricks facing different ways. Her concentration on the job in hand was complete.

The fencing off of spaces with an outline of bricks to form rooms, stables, &c., is an interesting and common feature of children's building, not unrelated to an appreciation of maps and plans. A very early and usual activity is the balancing of bricks on one another to form as high a structure as possible; the interest seems to be both in stability and in height. The fall gives as much satisfaction as the erection.

> Harry, not yet 3, danced for joy at his achievement of a tower of bricks, crowed with pleasure at its fall, and continued to rebuild and destroy not only for many hours but day after day.

Larger building efforts with boxes, barrels, and planks are also often an exercise in balance and firmness of structure, as well as in representation of familiar things.

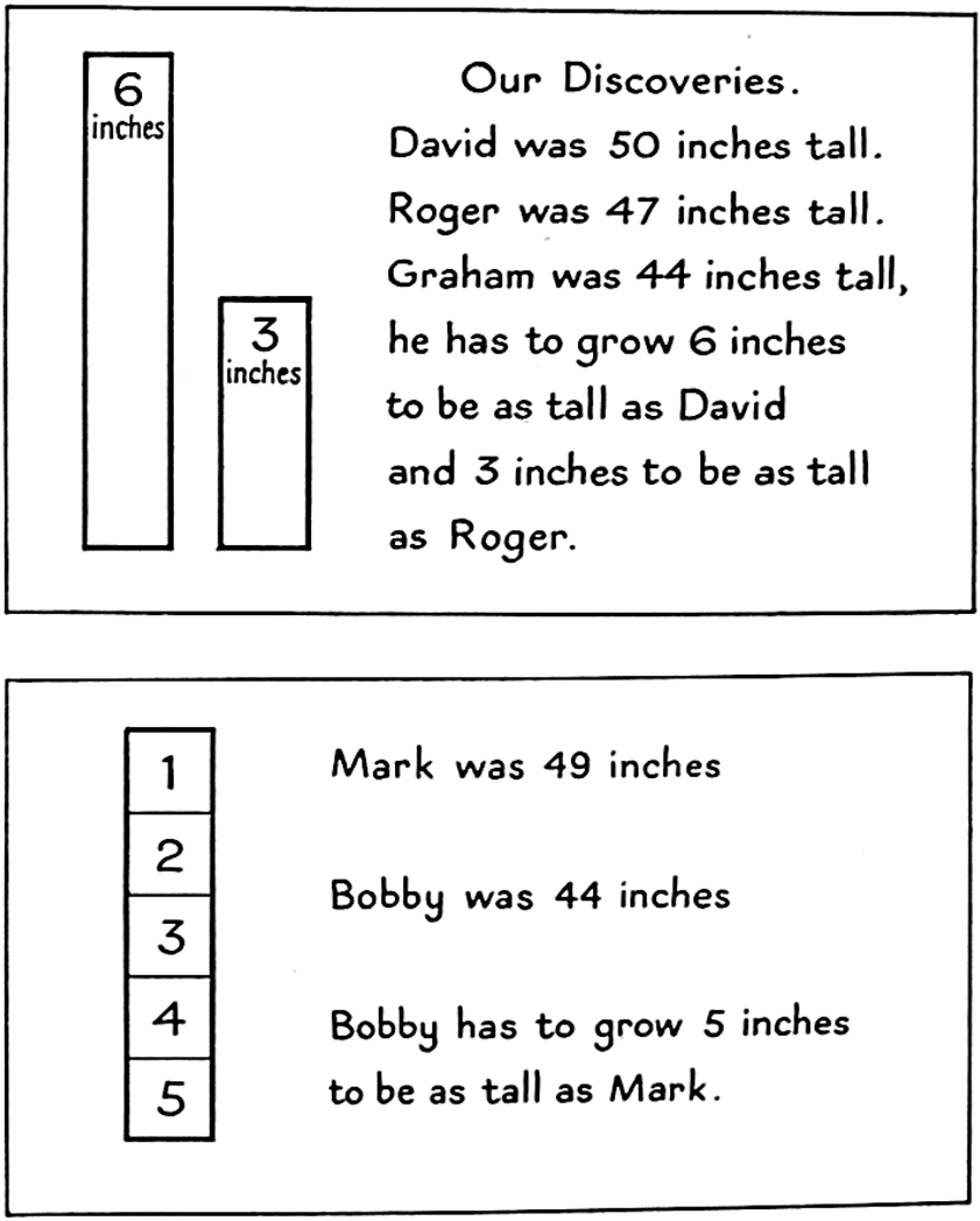
The knocking together at a woodwork table of pieces of wood to form objects of interest or use is a most fruitful experience. The 'aeroplane' made by two crossed pieces of wood shows clearly the maker's powers of judging both lengths and angles to give appropriate symmetry and proportions. The representation of a single form by nailed parts is more exacting than the quick building with bricks, and it may involve cutting off as well as choosing suitable pieces. At later stages usefulness may serve as a spur.

> Graham, at 7, wanted a tray to display specimens. He cut off one by one with care suitable edgings to his rectangular foundation for three of the sides, but his patience gave out for the fourth. Distaste, however, for an ill-looking job sent him back to remove the offending fourth edge, and he duly cut off the required length.

At a later stage he might well have forecast the need for two pairs of equal edges, but perhaps only have discovered by experiment that he had to make allowance for corners. Graham's 'measuring' of edges in this example only involves the matching of one length with another, but in fact this boy had previously used a yard-stick marked in inches, and he chose to make use of this in selecting and shortening his pieces of wood for the edges. He might have made a better job had he used the more straightforward method of matching. The carpenter who went down a Somersetshire street with his arms outstretched calling out, 'Get out of the way, I've the size of a pair of door jambs here!' probably made a good job of his work when he transferred his measurement to wood.

The measuring of lengths by adults and the use by them of the names of common units may be noticed and copied by children from quite an early age.

> Freddy, scarcely 4, was accustomed to refer to the large nails available in the nursery by their correct names. He came to ask for a nail more suited to the job in hand, saying, 'I've nothing but two- and three-inch nails; I want one of they little 'uns.'


6
inches
3
inches

Our Discoveries.

David was 50 inches tall.
Roger was 47 inches tall.
Graham was 44 inches tall,
he has to grow 6 inches
to be as tall as David
and 3 inches to be as tall
as Roger.

1
2
3
4
5

Mark was 49 inches

Bobby was 44 inches

Bobby has to grow 5 inches
to be as tall as Mark.

FIG. 13. Pages from News Book (measures were in actual size).

Again the noting of children's height, so dear to parents, stirs the children's interest in this personal measurement. Fig. 13 shows records made in a news book (the same book as that in Fig. 2), by the teacher of a class of 5-year-olds in a suburban area, of what the children were anxious to formulate about the measuring in school of their own heights.

This very early precise recording of measures does not necessarily mean that these children have not stages to be lived through which are of a simpler qualitative kind than these particular instances in which the precision is stimulated by environmental influences of home and school. There is, for instance, something different about a height and about a length measured in a horizontal plane, or about a girth. To come to terms with the first does not necessarily mean any command over the others.

> Kenneth, aged 5, ran from object to object with an ungraduated tape, stretching it out each time to match the different lengths and saying, 'I'm measuring, I'm measuring'.

He was in fact imitating measuring and experiencing the feeling through his actions of longer and shorter. At his stage visual discrimination of length or size would be more usual.

> Giles, a 2-year-old, remarked on seeing a St. Bernard dog, 'Oh Auntie, not one doggie, two doggies!'

He was making a visual judgement, and even though he used 'two' his judgement was qualitative. By *two* he meant larger rather than *twice* as large. Other examples of early visual discrimination, or the lack of it, are given later in Chapter XI in discussion of the approach to the idea of scale.

When a child passes from the general quantitative appreciation of 'more' and 'less' to a more definite description using numbers, he makes a very big step in understanding. He has, for example, in dealing with lengths to choose a particular length by means of which he will describe others. There are two ways in which children approach this notion of a unit, and these two approaches generally operate together. The one, which has already been illustrated, particularly by the story of Freddy and the two- and three-inch nails, is the acceptance through language and environment of adult conventional units and descriptions of size, with a gradually increasing appreciation of what the words imply; that is, quantitative language shares with all language imitative beginnings and growth of meaning with growth of experience. On the other hand, experience provides particular situations in which the choice of a unit, probably not a conventional one, arises from the nature of the activity. The painting or construction of a border pattern, for example, would involve a repeated element or unit of pattern. This unit must be chosen and placed on the required border, perhaps by folding an equivalent strip of paper into suitable units and using it as a marker. Again window-panes may serve as units in which to compare sizes: the large window is five panes across, the small one only three. The description of how long something is by the number of known pieces it contains is not a remote idea in practice.

Units based on the children's own hands, feet, or strides, involve activities which often appeal as a game. It soon emerges, for instance, that the larger the unit, the fewer the count for a given distance.

> Rosemary, at the age of 4, on her own initiative adjusted the unit from a large to a small hand span when she found the former gave her too few pieces for her purpose.

This informal use of 'natural' units seems a good beginning for use in school, but circumstances determine whether a child follows what appears to be a natural line of development; he may plunge into the middle of things by, for example, the discovery in Mother's work-basket of a measuring tape with its conventional units. Once the idea of counting up units large or small is established, the apparatus available should be suitable both for childish stages and for the purpose for which it is used. The tape from the sewing basket, however attractive as an adult tool, is too detailed in its marking, an awkward length for establishing standards, and an awkward tool for repeated use on a long line; it is a suitable tool for uses for which it is designed, and in particular for measuring girth.

Many teachers include among their available tools sets of laths in graduated lengths inch by inch from 1 to 12 inches, but some children will pass through this stage of discrimination without needing this special apparatus. The most useful tools at early stages are simple foot- and yard-sticks, the former marked clearly in whole inches, and among the latter some unmarked, but more graduated in feet or inches or in feet and inches. Refinements in the use of length-measuring instruments, though they may sometimes be employed by quite young children, are left for later discussion. One example of the use of an unusual unit will take this section about length to a suitable conclusion.

> Some 6-year-old children were measuring the length of their feet and then making heel to toe measures of a shortish length. George went off to measure in this way the length of a grass edging. His foot was 7 inches long, and he was in trouble when he found his count was 14 of his own feet. The subsequent counting of the total number of inches using the familiar apparatus from the classroom, viz. little sticks laid out in groups of seven, took the activity far from its original purpose; but the readiness with which the boy converted his problem into a count of numbers of sevens showed considerable grasp of what he was about.

Measuring of any kind involves this key idea of a unit and its choice. With other measures, as with length, imitation of adults and the chance contacts of environment may upset what might be expected to be an orderly sequence of development, proceeding through comprehension of what is to be measured to means of measuring it. Probably time affords the most striking example; children use words connected with the passage of time when they have only hazy notions of what these words signify. 'I'm four' or 'I'm five' is a childish statement with little meaning except for some feeling of status in comparison with others; but 'I'm six' is often short for 'I'm six years old', with some idea that the year is the wait from birthday to birthday, just as Christmas is a birthday and we have to wait twelve months for its return. Similarly, references to hours and minutes and even telling the time, especially telling particular times associated with routines of the day, are at first conventions of speech rather than indications of understanding.

Again, a child who sees weighing and measuring in the kitchen or the shop may use the words 'pound', 'ounce', or 'pint' with only a shallow appreciation of what is going on; but there is a growing familiarity which increases his readiness to give the ideas precision by personal experiment. If his mind is not ripe, his actions will be only superficial mime.

Weight in particular is an idea approached rather slowly. Personal discrimination between weights is never very easy; unlike visual discriminations, comparisons cannot encompass several items but must be made pair by pair, and the lifting of two articles together in different hands is not a safe guide. Beam balances of the simple see-saw variety, not too sensitive and with large equal scale pans, are commonly used by children of four, five, and six, both in their imaginative play and in their experiments with different materials. If these materials include those which are widely different in density, for example glass marbles and dried peas, the possibility of the balancing of a small bulk with a large one becomes noticeable. Collections of small objects of equal size and weight, possibly even objects like chestnuts of approximately equal weight, suggest the counting of the contents of the scale pans, and so approaching the idea of using a unit of weight.

An extract from a news book made for children of 5 years of age reads:

> 'Christine discovered that 12 one-inch nails weighed the same as a three-inch nail.'

Balances used in shopping and home play keep what is done in touch with everyday experience. As soon as a unit of weight is brought into use it is important to have a number of these units equal in appearance as well as weight; if they are 'ounces', there should be at least sixteen of them. The familiar grocer's pyramid belongs to a much later stage. So, too, does the grocer's balance with its unequal pans and intricately constructed arms. A spring balance with its scale or dial has the merit that two units moves the pointer twice as far as one. The see-saw itself gives children a means of experiment with their own weights.

Richard, aged about 7½ years, had gained his early ideas of weight through shopping and other such activities in school, but he extended them through the record he made of the progress of his much-loved baby sister. He kept a daily diary over many months. Extracts are quoted, most of them selected for records of weight:

May 26. We Have a New BaBy girl and Her Name is Jose-PHINE.
27. She weighed 7 lbs. 4 ozs. She was born on May tHe 14th.
29. SHe is very good and sleeps a lot.
30. SHe now weighs 8 lbs. 4 ozs.
June 2. SHe is the first girl in our family after three Boys.
8. Doctor came to see BaBy's bad eye again toDay.
9. BaBy gained ¾ lbs. SHe is now 9 lbs.
13. My BaBy smiled at me today.
July 11. Mummy took BaBy to Clinic today. She gained 6½ ozs.
12. Josephine weighes 9 lbs. 10½ ozs.
14. Bady is 2 months old today.
26. BaBy weighed 9 lbs. 15 ozs.
Aug. 3. Josephine haves Bengers in her milk now.
14. Baby Kicks all her clothes off of her now.
15. Bady is 3 months old now.
16. BaBy now weighes 10 lb. 14 oz.
21. Bady laughes at people now.
30. BaBy now weighes 11 lbs. 12 ozs. day.
Sep. 6. My baby now weighs 12 lbs. 5½ ozs.
7. I love my baby Sister.

Capacity has already been discussed at some length. Activities which almost amount to measuring frequently occur in play, as, for example, the teapot in the home corner which holds so many cupfuls; but the magic changes of shape which makes a cupful so different when it is poured into a glass or spilled on the floor may delay the appreciation of a unit of capacity. This does not mean that children cannot or should not experiment with liquid measures. Once children have come to terms with change of shape and have mastered the practical problems and distractions of pouring and spilling, they can experiment with liquids with the great advantage that the liquids can be broken up and put together at will.

> Robert, a rather able $5\frac{1}{2}$-year-old boy, asked for water when he had a choice of what he would like to do. He was given two buckets, some water, and two old-fashioned measures which he knew to be a pint and a half-pint. In self-set imitation of some more formal exercise he filled his pint measure using the half-pint and transferred water from one bucket to the other. When asked 'How many of those have you?', he replied 'I don't know', but much later he called the questioner to him and said 'I have 8 pints and 16 half-pints'.

In this correct double count there is evidence of experimental deftness and persistence as well as number skill.

A practical point to be emphasized is that it is the child's whole day which, at these early stages, is full of opportunity for pre-mathematical experience. This implies that the teacher has a continuing part to play in watching over progress in mathematical development.

She must in the first place see to it that material opportunities are there in plenty for the children. In particular she must provide a sufficient variety and grading in size and shape of all those subsidiary materials which have been called tool materials and which include natural and household objects as well as simple instruments for construction and measuring.

Then she must herself recognize in activities values of which the children themselves are hardly aware, and she must restrain herself from, for example, forcing a quantitative stage on individuals for whom the activity is still properly only a qualitative one. On the other hand, she must recognize the symptoms of readiness for new fields to conquer and by approval, suggestion, or the provision of new material, stimulate advance.

Many schools possess amongst their equipment graded apparatus specially designed to bring out mathematical properties such as graded sizes and different shapes. Many of these are of only very limited use. They just mark a stage of appreciation or serve as a clarification of more general experience. The children's interest in them is a good guide to their usefulness at any particular time. Few last an individual for long, but the teacher may use them for a quick stocktaking of stages of growth. There are, however, one or two types of apparatus which are of special and more lasting value. The long series of similar shapes which can be packed one within the other, e.g. nesting cubes or cylindrical tins lacking a top or other nesting shapes which have to be taken apart, are typical of important mathematical properties. For instance, the fifth in size contains the four smaller shapes. It is interesting that this valuable type of apparatus has greater attraction and more varied use than most specialized apparatus. Again certain special toys offer great opportunity. Jig-saws clearly exercise children's judgement of shape and size. Constructional toys do this also and in addition they may offer some mechanical experience. Any toy with moving parts may have values both spacial and mechanical. Indeed, sufficient attention is seldom given to children's great interest in patterns of movement.

Finally, it is the teacher who is the main school source of the language environment in which the children live. It is important that while not instructing them in the use of the simple technical terms in ordinary speech, such as *circle*, *ring*, *oval*, *corner*, *curve*, and so on, she should herself use these words correctly and not hesitate to use them when the context makes clear to the children what she means. 'I should use two-inch nails from this box, the three-inch nail you have is too long' may have been the source of Freddy's clear statement later about the nails he wants. 'If you hold the string tightly we shall know how much we need to go straight across from corner to corner'. 'This leaf has a pointed end. Can you find one with a curved end?' 'Hold the yard-stick upright when you are finding this height and not leaning to one side'. 'Do you think that square piece of wood would make a good tray or would an oblong piece, like this, be better?'

Even though many of the words are not echoed in the children's own speech, every time a word is used correctly and significantly in their hearing it gathers meaning and increases awareness. It is also much less likely to be used incorrectly later on. For example, the word 'straight' is frequently misused by much older children just because they have no alternatives for words of direction such as 'upright' and because 'straight' itself has few associations except the colloquial 'straight up'. It is the teacher who uses the right word in the most natural and unforced way who is most likely to influence the growth of the child's understanding and vocabulary.

IV

TRANSITION: A CRITICAL DIGRESSION

When children have learned to count and have had some opportunity of dealing with practical situations by counting either objects themselves or sticks and counters representing them, it is quite a common practice in schools to introduce the children more or less abruptly to signs and symbols. These rightly belong to a world of numbers, but are often associated in the teaching with the manipulation of varying numbers of counters rather than with real experience or with number relations.

Thus 4+3 may be introduced as a direction to take 4 counters and 3 counters, to set these together in a heap, row, or pattern, and then to count the whole, or to count on from the 4 heap till the 3 heap is exhausted, and write down the result

$$4+3=7 \qquad \text{or} \qquad \begin{array}{l} 4+ \\ \underline{3} \\ 7 \end{array}$$

This, as set out, is an extreme case of arbitrary introduction without relation to experience other than counters. It is, however, all that effectively survives a great deal of teaching of 'little sums', when these are treated *at very early stages* as ends in themselves. This sort of thing may occur even before figures are firmly established as standing for the number in a group. The introduction of the other signs, −, ×, and ÷, is in a similar way often premature, detached and, therefore, meaningless. To put it more positively, it is rare to find the introduction of, for example, minus, arising from the recording of the number content of some worth-while situation, preferably some situation which was seen to have a number resemblance to previous experience. To take as a part analogy the introduction of a new word in the development of language, it is as though the word 'divided' was introduced for the first time in the sense it is used when we say 'We are not divided' or 'A house divided against itself', instead of first introducing it with its material sense of 'A loaf divided up amongst us', until it is quite familiar. Yet the way in which mathematical symbols are introduced at an early stage often suggests that the abstract can be mastered without much reference to a full experience of ordinary concrete usages. The mathematical symbols under consideration often collect only the thinnest shell of meaning from their connexion with the typical experience of counting counters. The counters, perhaps, have themselves only flimsy associations with living and doing gained from their occasional use as an aid to the solution of practical problems.

The long transition from experiment to reckoning and from particular experiences to general statements of their number content is a critical one in determining the health of arithmetical thought. The treatment of the introduction to arithmetic in the years from six to at least nine for normal children and longer for the less able therefore requires great care and skill.

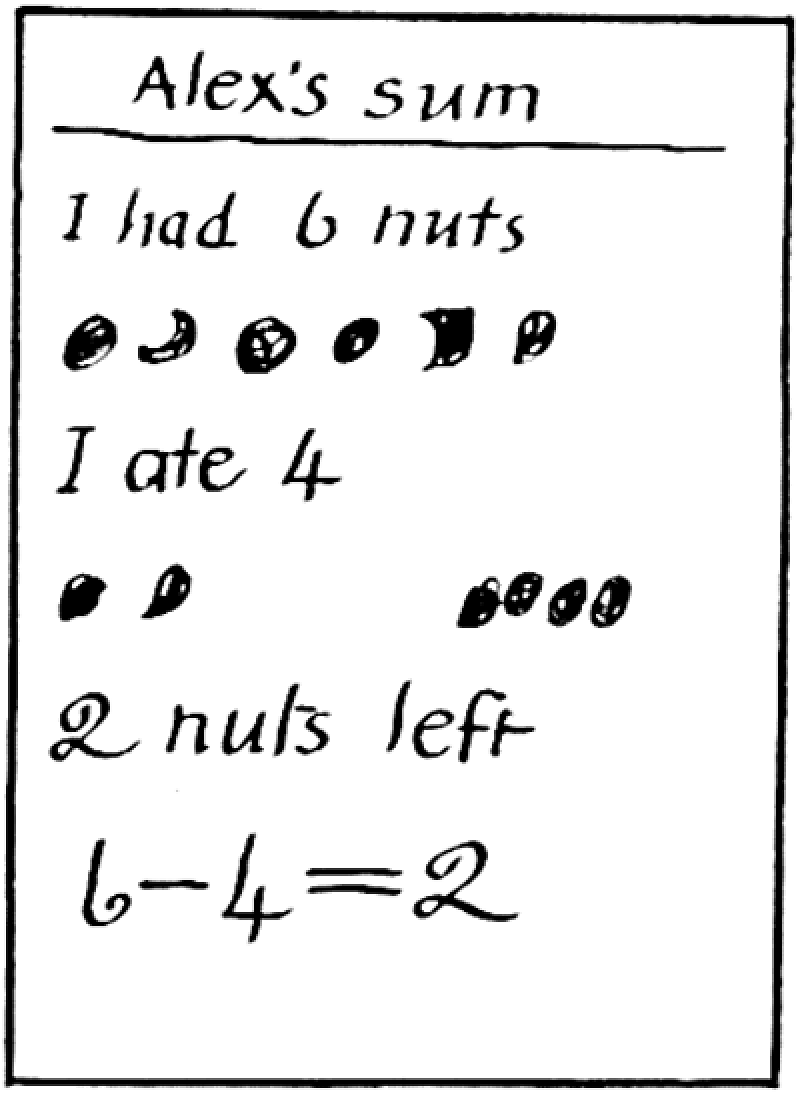
Because of the nature of common practice in schools, with its pressure to get on with reckoning as soon as possible, it is difficult to illustrate an alternative programme very fully. In earlier chapters illustrations of what children do and say have been in almost every case positive and constructive. From the time that 'little sums' appear, whether it be at 5, 6, or 7 years of age and whether the child be able, average, or dull, it becomes increasingly difficult to find illustrations of well-balanced development. Illustrations, therefore, tend to become analytical or definitely destructive, and suggestions about what children can do become more tentative because they have less positive experience behind them. There is, however, much scope for the adult to become better informed about what, in fact, is happening and about what is involved in the subject, even if suggestions about how it may be learned still remain more a matter of conjecture; and assumptions have to be made about the propriety of learning what is very scantily understood.

Justification for suggesting that even at 6 years of age all is far from well with common practice can be found from a diagnosis of what in fact children understand about what they are doing at 7, 8, and 9. Here is a diagnostic sequence as illustration. This sequence is based on a remedial game, seen many years ago, which was being used to help some older infants whose syllabus included multiplication and division signs but who were unable to use multiplication and division in solving little verbal problems. It is, of course, normal for infant children to solve their problems by crude practical means, including a recognition of the answer from mere visualization of the situation, and it would be most unusual for them to have enough experience and number knowledge to prevent the early learning of a multiplication table from becoming a feat of memory isolated from experience. The premature teaching of multiplication and division, with the signs × and ÷, often amounts to little more than numerical definition; there may be the support of some manipulation of counters, but the efficacy of this largely depends on the background of experience, practical and numerical, which the child himself brings to learning. Figs. 14 and 15 show two examples of the game referred to above, which the children called 'I saw a sum'. They made up their own story sums from something seen, wrote about them in words with any necessary help, drew pictures to illustrate them and finally decided on the 'little sums'[1] with appropriate signs which belonged to the stories.

A primitive solution of the problem is given by the pictorial element in the statement in Fig. 14 or 15. The association of this with a symbolic statement is a step towards making meaningful signs already taught, but it was not a natural order of learning, though helpful in unnatural circumstances. The 'sums seen' by the children did not always conform to expectation. For instance, a 'sum was to be seen' in a wall illustration consisting of fifty

---

[1] The word 'sum' is so commonly used by children for any calculation that it cannot always be avoided in inappropriate circumstances in quotations.



Alex's sum

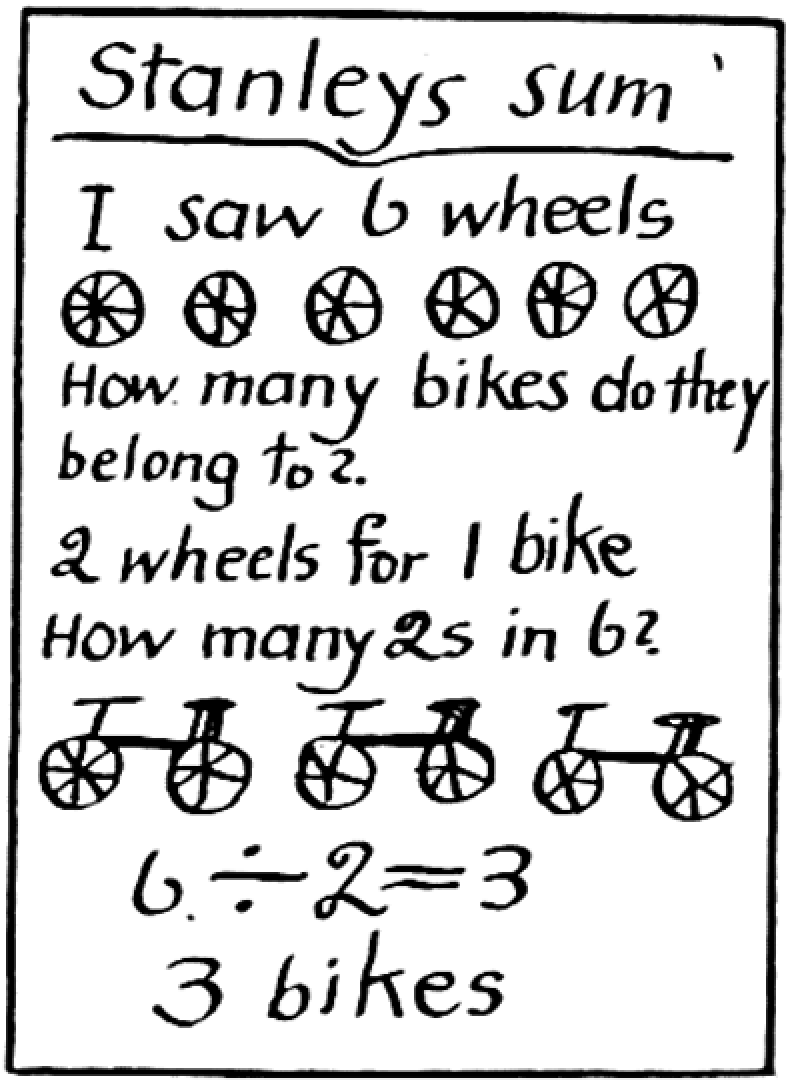
I had 6 nuts

I ate 4

2 nuts left

6−4=2

FIG. 14.



Stanleys sum '

I saw 6 wheels

How many bikes do they belong to?

2 wheels for 1 bike

How many 2s in 6?

6÷2=3

3 bikes

FIG. 15.

cigarette cards mounted in five rows of ten; instead of the expected response leading to 10×5 = 50, a boy said 'There's two and two and one in the middle', referring to the symmetry of the 5 rows. Although this statement probably echoed some previous training in the domino arrangement of 5 dots, it is an interesting adaptation to the symmetry of rows and indicates an interest in pattern rather than in 'sums'.

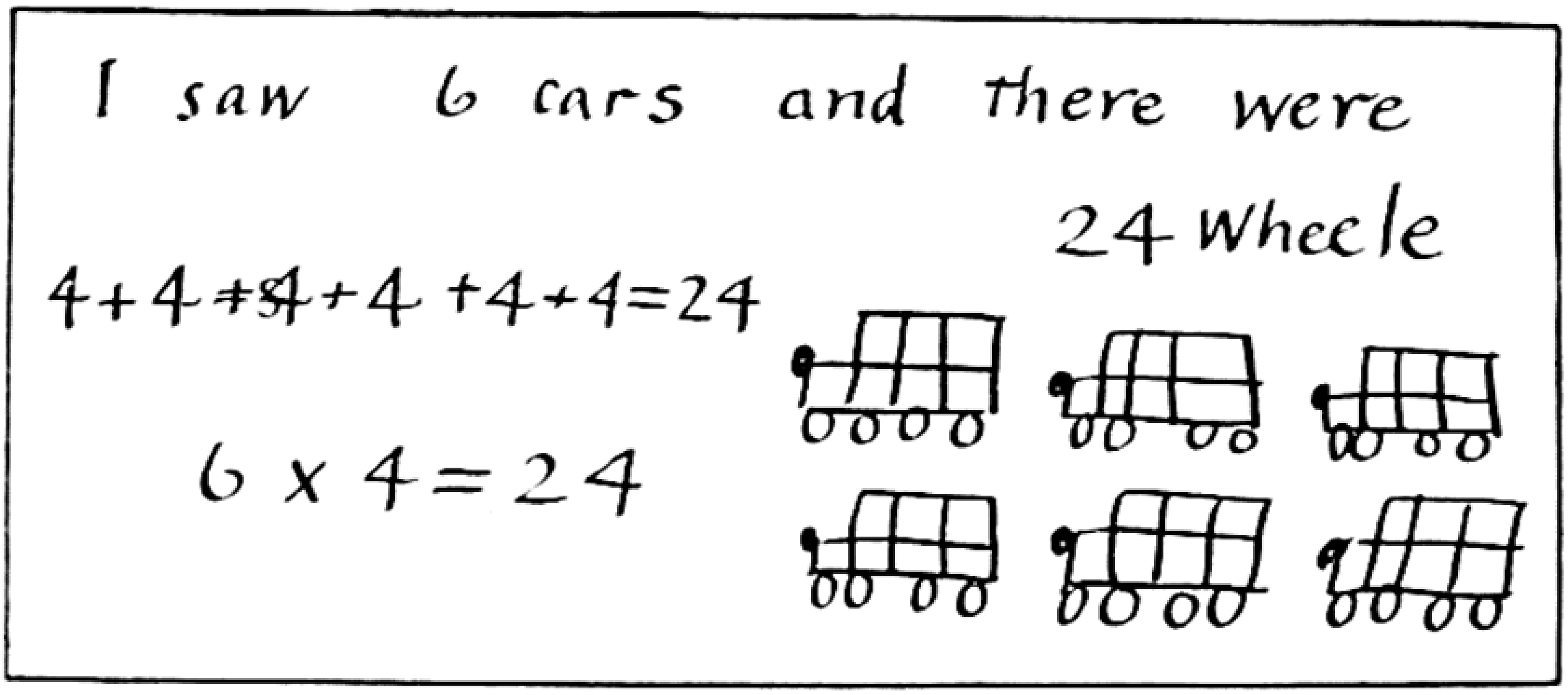
The substance of Figs. 14 and 15 has often been described to children of 7 and 8 years of age who have been taught the four signs, +, −, ×, and ÷, in order to encourage them to make up similar stories for themselves and so reveal something of their understanding of the purpose of the reckoning which they are doing. Fig. 14, or a similar story, can be used to explain the game 'I saw a sum'. In discussing the 6 wheels of Fig. 15 it is normal to have a ready answer that there are 3 bikes. When the wheels and the bikes have been drawn, there is also ready agreement that 2 will probably come into the final statement because there are 2 wheels to a bike. Children's attempts to give this final statement show wide variation. It is generally suggested to them that though one particular sign is the best, others can be used quite sensibly to connect the numbers belonging to the story. An ingenious boy once wrote $6-3=3$. Protest that this was true but did not belong to the story was refuted by the statement, 'But it does, there were 3 front wheels and 3 back'.

At 7 and 8 years of age $2+2+2=6$ is quite a normal answer and $2\times3=6$ is good. Nil results of course occur, and it is quite common to have, in spite of the drawing of the wheels and the bikes, the following nonsense statement:

I saw 6 wheels
There were 3 bikes
2 wheels to 1 bike
$6+3+2=11$

Abler children of 8 years of age, especially if they have division fresh in mind, may give the correct $6\div2=3$, which they may state thus, $\begin{array}{r|l} 2 & \underline{6.} \\ & 3 \end{array}$ Interchange of divisor and quotient, $\begin{array}{r|l} 3 & \underline{6,} \\ & 2 \end{array}$ is not uncommon. All these variants, together with the completing statement 'It is how many twos in six', have been obtained from a single class of children of between 7 and 9 years of age.

The only justification for asking children of this age a question like this is that their computation practice suggests that they have long passed the stage of knowing what is the meaning of the signs and the purpose of tables. The stories of Figs. 14 and 15 also form an introduction to asking children to make up their own problems and show solutions and statements. A number of such efforts are shown in Figs. 16–25; Fig. 11 of Chapter II was also obtained in this way. Most of these examples come from children who were accustomed to lively oral problems, and these have probably influenced the children's choice of subject-matter. Most of the examples, too, show the children to be at least in touch with what they are doing. This is by no means always the case; the example above shows even + used in a meaningless way, to obtain a number 11 which has nothing to do with the problem.

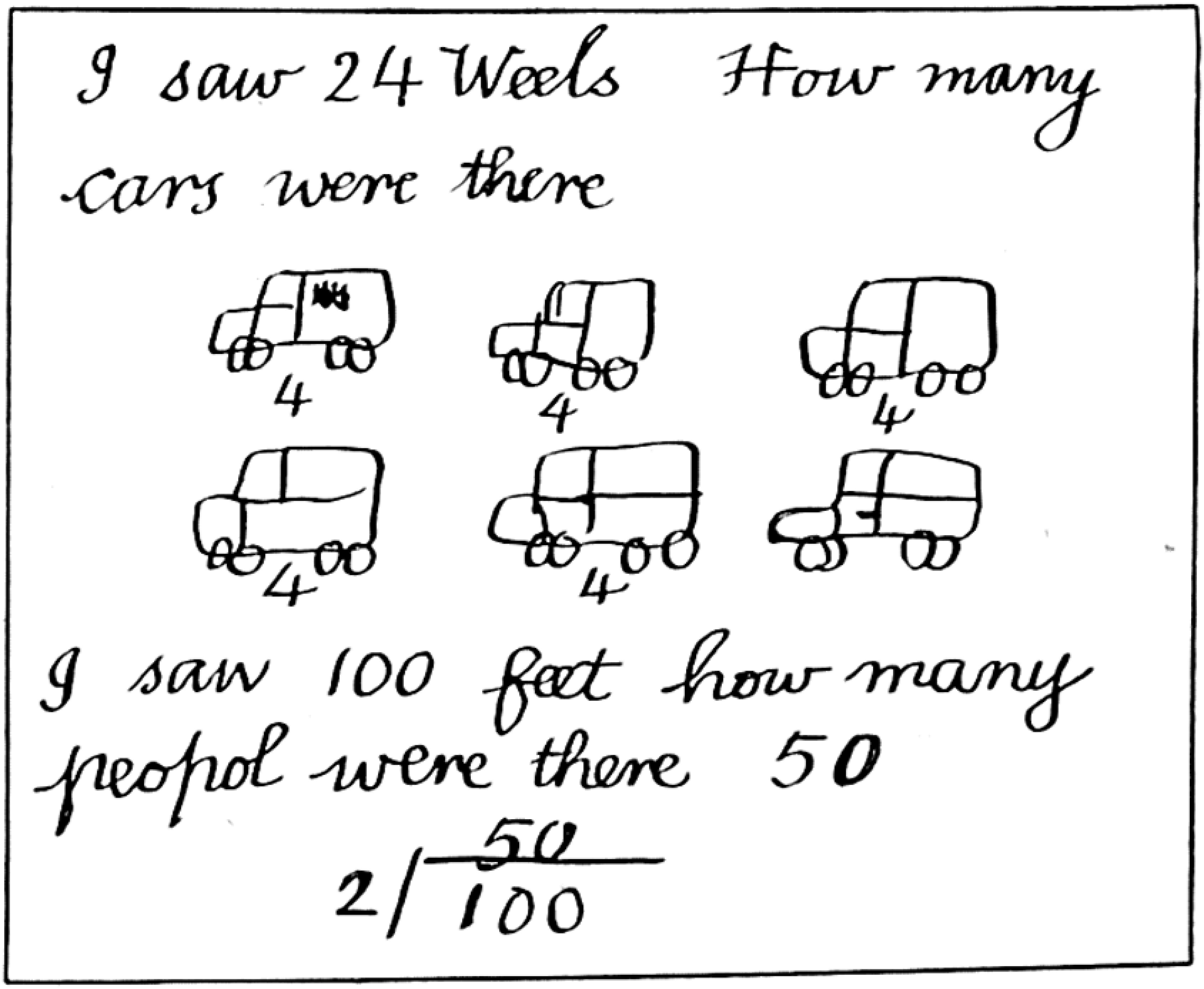


I saw 6 cars and there were 24 Wheele

4+4+4+4+4+4=24

6 x 4 = 24

Fig. 16.

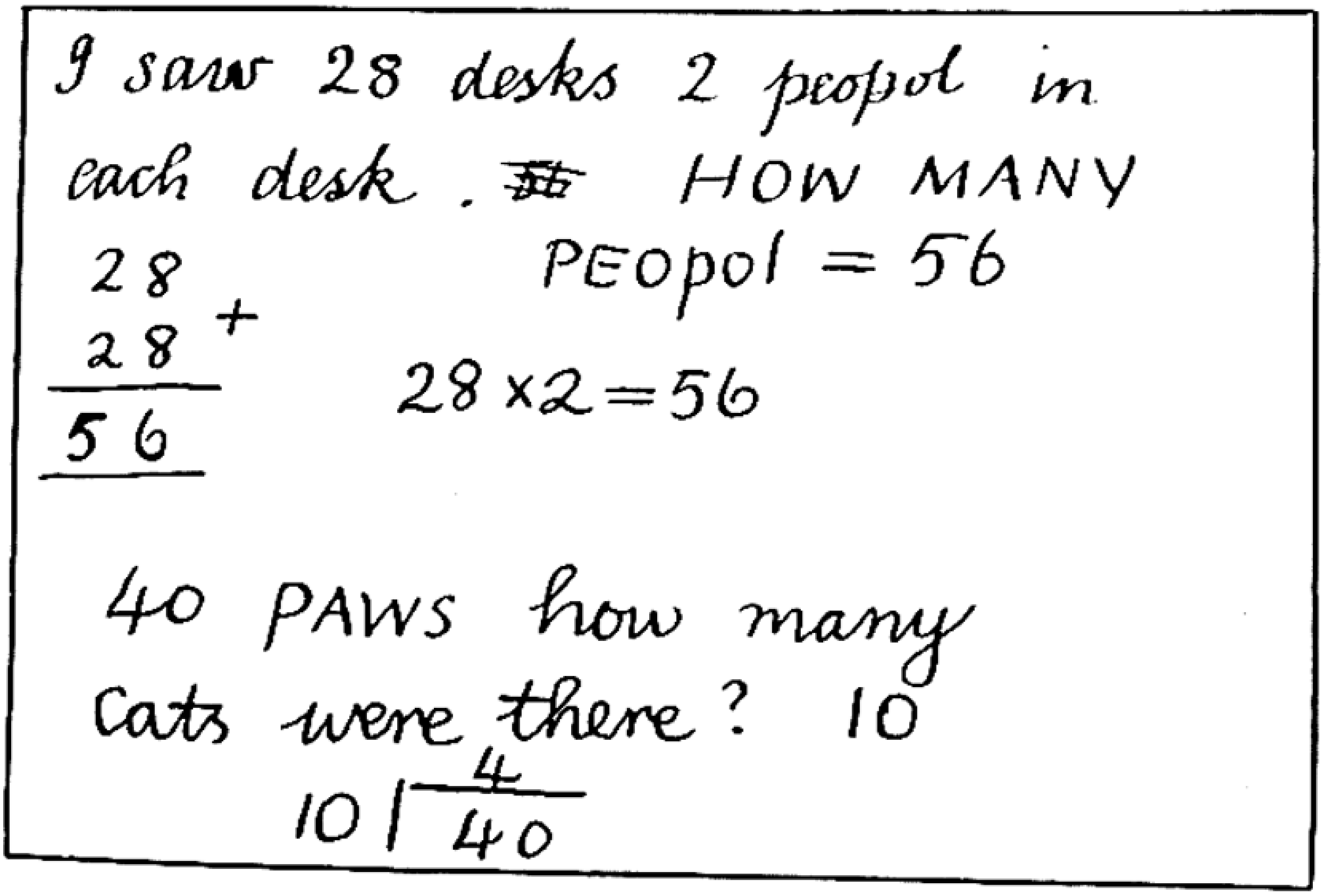


I saw 24 Weels How many cars were there

I saw 100 feet how many peopol were there 50

$$\begin{array}{r}50\\2\overline{)100}\end{array}$$

Fig. 17.



I saw 28 desks 2 peopol in each desk. HOW MANY PEOPOL = 56

$$\begin{array}{r} 28 \\ +\,28 \\ \hline 56 \end{array}$$

$28 \times 2 = 56$

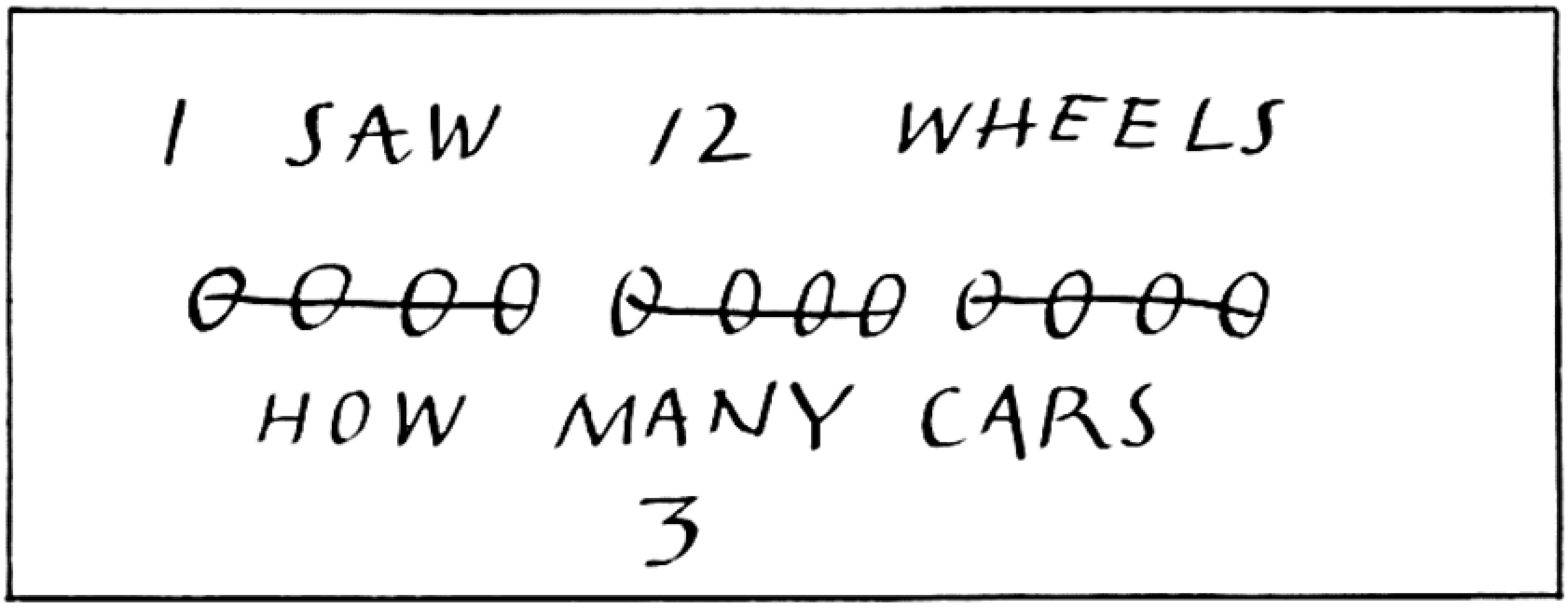
40 PAWS how many Cats were there? 10

$$10\overset{4}{\overline{)40}}$$

FIG. 18.

The boy whose work is shown in Fig. 16 was given some help. He was encouraged to write down how he found the 24 and began to write $4+4=8$. Further encouragement to 'write them all' produced the line of fours. It was then suggested that there was 'a shorter way of writing it', and the $6\times 4=24$ was written with evident pleasure.

The boy of Fig. 17 draws cars and writes 4 against each until he has exhausted the 24, when he neglects to write the final 4. He does not summarize by stating an answer, but hurries on to a similar type of problem which he does by an arithmetical method without illustration. Fig. 18 shows first an interesting variant on Fig. 17, in which duplication is preferred to counting twos and then replaced by the alternative of multiplication by 2. A problem soluble by division follows but the working shows artificiality as the answer to the problem is used as divisor. Fig. 19 shows a competent piece of practical work; diagrams of car wheels are

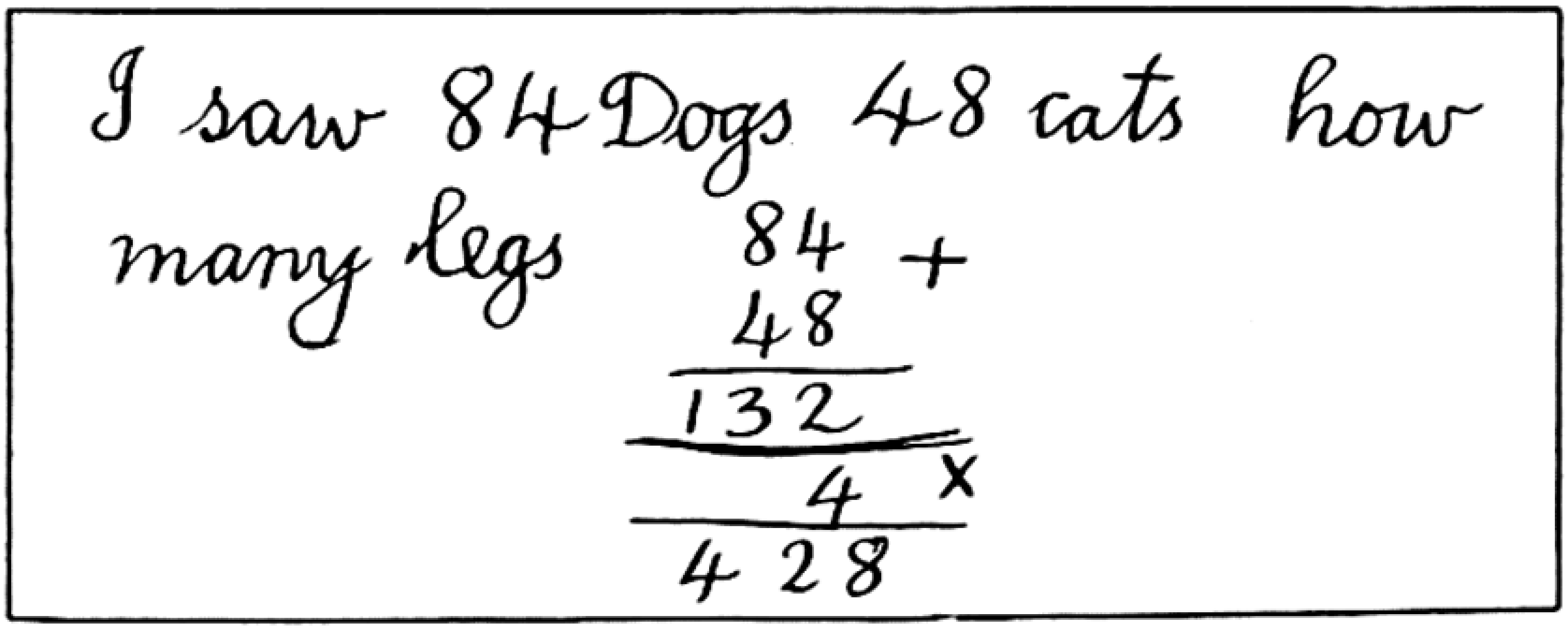


I SAW 12 WHEELS

HOW MANY CARS

3

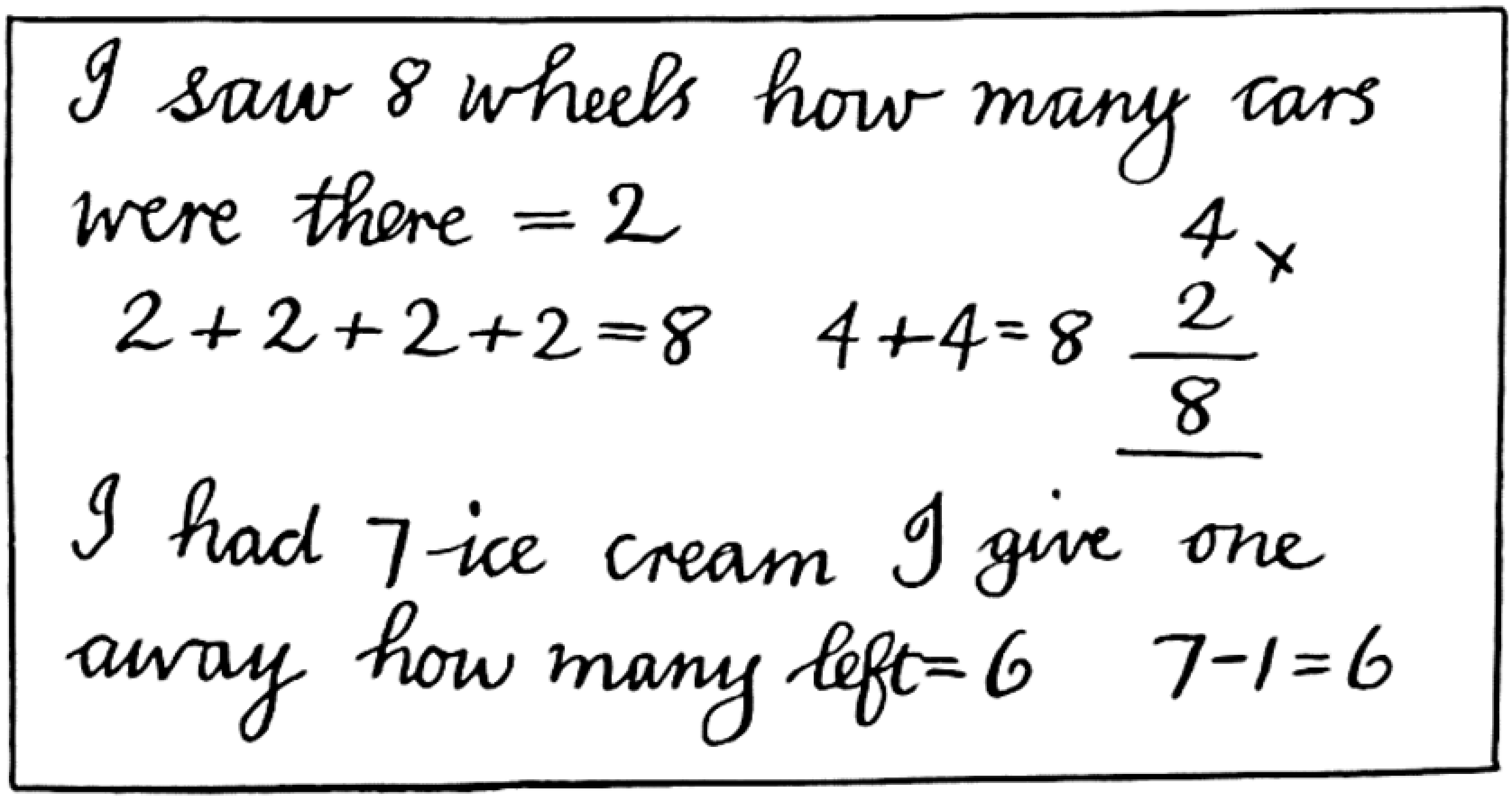
FIG. 19.



I saw 84 Dogs 48 cats how many legs

$$\begin{array}{r} 84 \; + \\ 48 \\ \hline 132 \\ \hline 4 \; \times \\ \hline 428 \end{array}$$

FIG. 20.



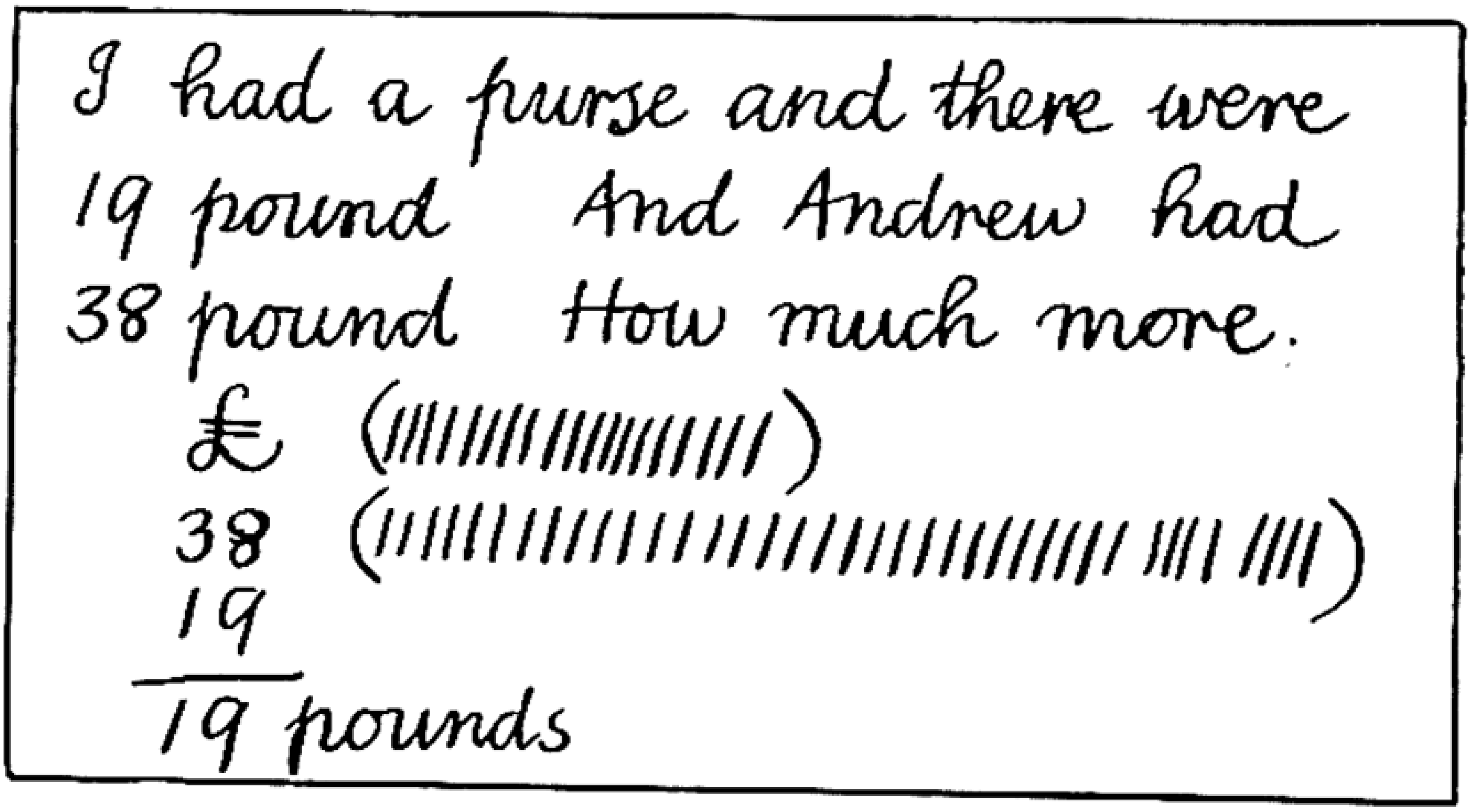
I saw 8 wheels how many cars were there = 2

2+2+2+2=8   4+4=8

$$\begin{array}{r} 4 \; \times \\ 2 \\ \hline 8 \\ \hline \end{array}$$

I had 7 ice cream I give one away how many left = 6   7-1=6

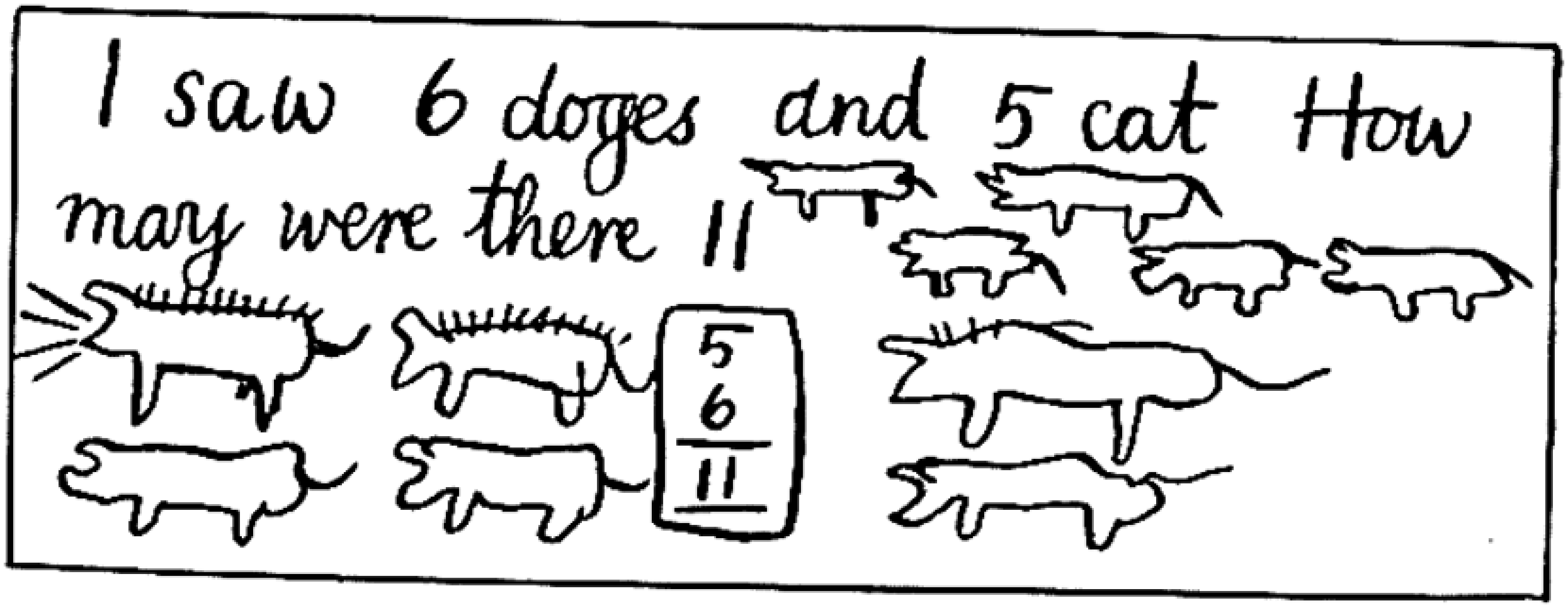
FIG. 21.



I had a purse and there were
19 pound And Andrew had
38 pound How much more.

£ (||||||||||||||||||||)
38 (||||||||||||||||||||||||||||||||||||||)
19
19 pounds

FIG. 22.

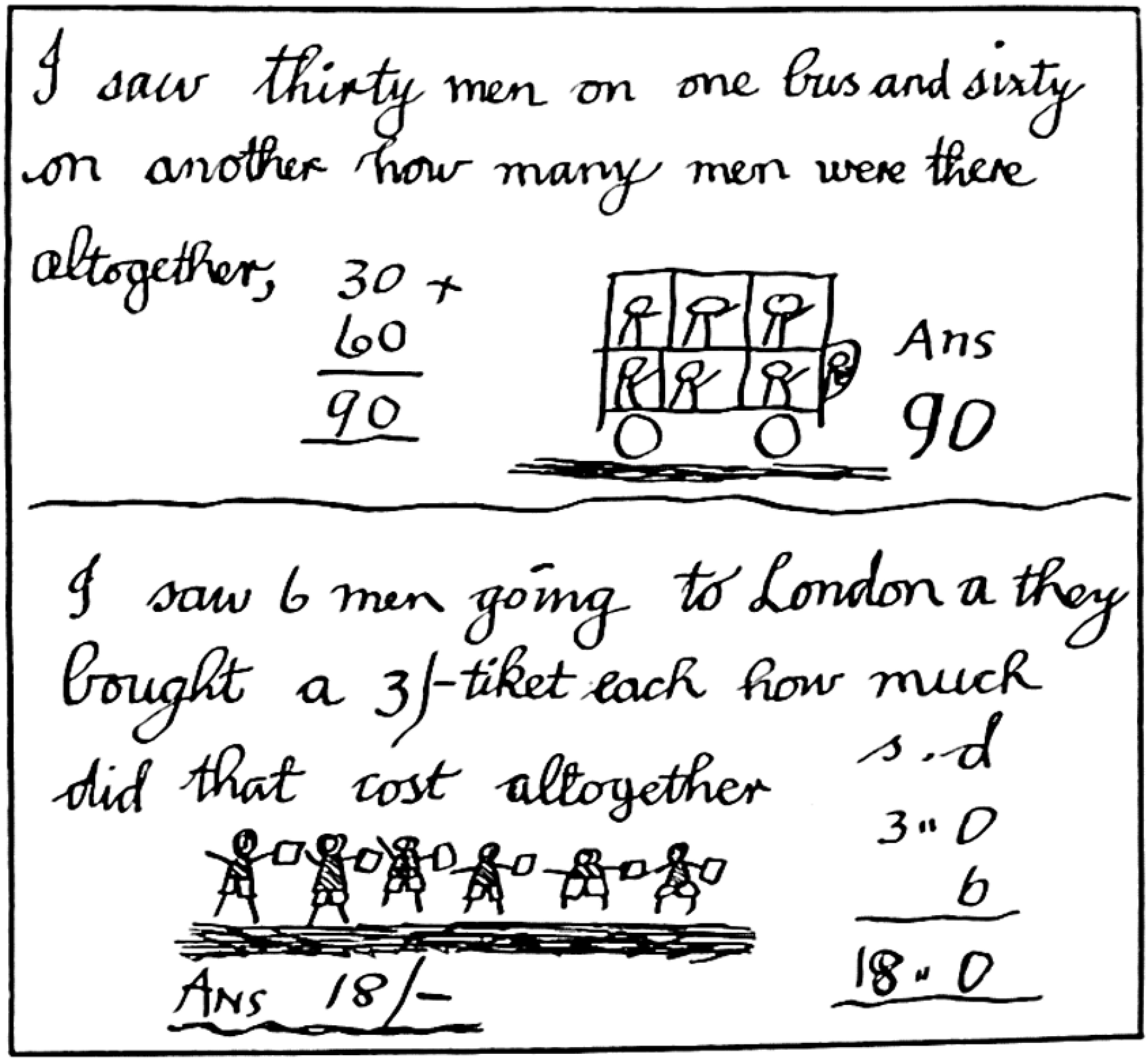


I saw 6 doges and 5 cat How
may were there 11

5
6
11

FIG. 23.

used instead of the pictures of cars in the similar problem of Fig. 17. Fig. 20 is by a boy very well advanced who first finds the number of four-legged animals and then attempts a sizeable multiplication. Four of the five remaining children are at early stages. Fig. 22 shows crudities of method of computation. In a class of wide range of ability there would probably be a large group whose efforts were comparable with or much below those shown in Figs. 21 and 23.



I saw thirty men on one bus and sixty on another how many men were there altogether,

30 +
60
90

Ans
90

I saw 6 men going to London a they bought a 3/-tiket each how much did that cost altogether

s . d
3 " 0
6
18 " 0

Ans 18/-

FIG. 24.

Fig. 24 shows larger numbers, money, and a notably different approach from an able girl. The illustrations in this are pictorial, not diagrammatic. Even the six men in the second part are not particularly helpful in deciding what must be done.

Fig. 25 shows the effort of a very backward boy of 9 in a special class. He was doing mechanical arithmetic with largish numbers and overheard a conversation about the possibility of making arithmetic more meaningful, in which the 'I saw a sum' stories were described together with other more practical means of approach. His effort was spontaneous, though he may have had

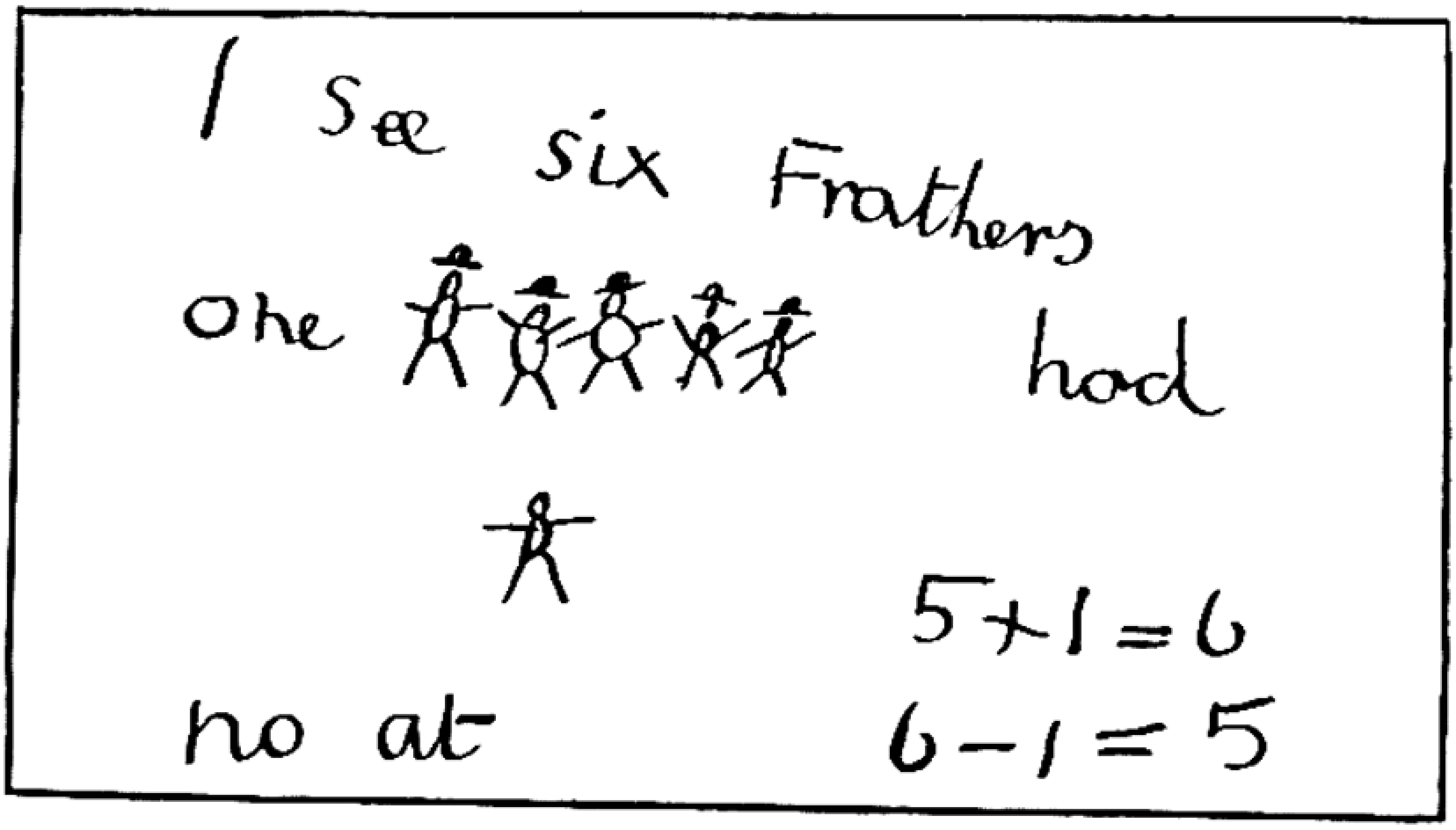


FIG. 25. By a very backward boy of nine.

some help with the alternative statement 6—1 = 5 for the 6 fathers, one of whom had no hat.

To summarize, many children of 7 and 8 years of age are taught the multiplication sign, and generally the division sign also. They generally learn the multiplication tables and do some multiplication of numbers with 2 digits. Short division of numbers with 2 digits is often included. This would describe the range of reckoning of the children whose problems are shown in Figs. 16–24, and on the whole this is a favourable sample of such responses to the diagnostic procedure described. Nevertheless, only a few of the examples show any confidence in the multiplication and division signs. The quite able children, whose work is shown in Figs. 16, 17, and 18, gained some confidence actually during the diagnostic activity. For many children—for example, the child producing Fig. 23 or 2+2+2 = 6 as the statement for the bikes problem—addition is the only process with meaning and multiplication is reckoning without use. This is to be expected among children who learn the procedures of reckoning on the basis of the experience acquired by 7 years of age. Able children can supplement what they are taught more easily than others as they go along. Probably the greatest loss to them, if any, is a loss of initiative at the outset if some symbol, for example ×, is more or less arbitrarily defined for them and only given purpose and significance later.

Other ways of approaching the stage of formal reckoning are discussed in the chapters which follow. This chapter only offers a set of fairly normal reactions at this most important stage, 7 and 8 years of age, to a teaching of reckoning of a usual kind and well up to the average in quality.

Any criticism of contemporary practice is open to the objection that no school can fairly be called typical of its time; the range of outlook as well as of material circumstances is too great. The school in which children are silent prisoners in desks exists alongside the school which believes in co-operative effort and freedom of movement, as do schools giving class tuition and those in which every effort is made to cope with the needs of individuals. The trends of educational thought can, however, be typical of their time. There is undoubtedly increase in the number of primary schools which expect the children to talk sensibly to one another, which give opportunity for direct experience and experiment, and which take more heed of individual differences. There is, however, no subject which lags farther behind its fellows in this than mathematics, and yet there seems no subject in which individual differences, stressed alike by teachers and by psychological research workers, are more likely to call for consideration. It is a frightening thought that in a subject which has so much to do with how we think as well as what we think, so much effort should be spent on learning so-called number facts and rules with so little differentiation based on knowledge of the individual learner. There is, too, no subject in which conventional approaches and schemes of work more often take the place of understanding and critical selection by teachers.

V

PRACTICAL BACKGROUND TO ARITHMETICAL PROCESSES

(a) AND-ING, COMPARING, TAKING AWAY

COUNTING in ones is an accomplishment which will serve a very great number of the ordinary purposes of life which involve reckoning; it will serve most of those of interest to children, especially if some tallying material such as sticks, counters, or abacus, is available to help 'keep count' when numbers are large or the grouping involved at all complicated. For instance, the 14 sevens involved when George of Chapter III used his 7-inch foot to measure the length of a grass border was easily dealt with when the counting material, sticks, was brought out to help.

Using counting, however, has its stages. Suppose a child wants to find out how many marbles he has; he counts them one by one and reaches the total six. He then notices another marble in a corner. There is a stage at which, having noted 6, he will not say 'and another one 7', but will go back and count the six over again continuing to seven. Again, supposing he has three of his marbles in his hand and recognizes them as three, at one stage he will accept 3 and go on counting 4, 5, 6, 7, but at an earlier stage he will put all together and count them from 1 to 7.

So long as only one of the groups concerned in the counting, e.g. 3 marbles in the hand, is recognized as a numbered group the procedure differs very little from a straightforward count, but if two or more groups are noted separately and then counted, 3 marbles in the hand *and* 4 on the table, or 3 spots on one side of the domino *and* 4 on the other, or 3 pennies for the ball *and* 4 pennies for the doll, the resulting 7, even if obtained by counting ones, has the two shapes—that of seven ones and that of a group of three *and* a group of four. This is most perfectly shown in the story of Rosemary in Chapter I where the flower heads of a

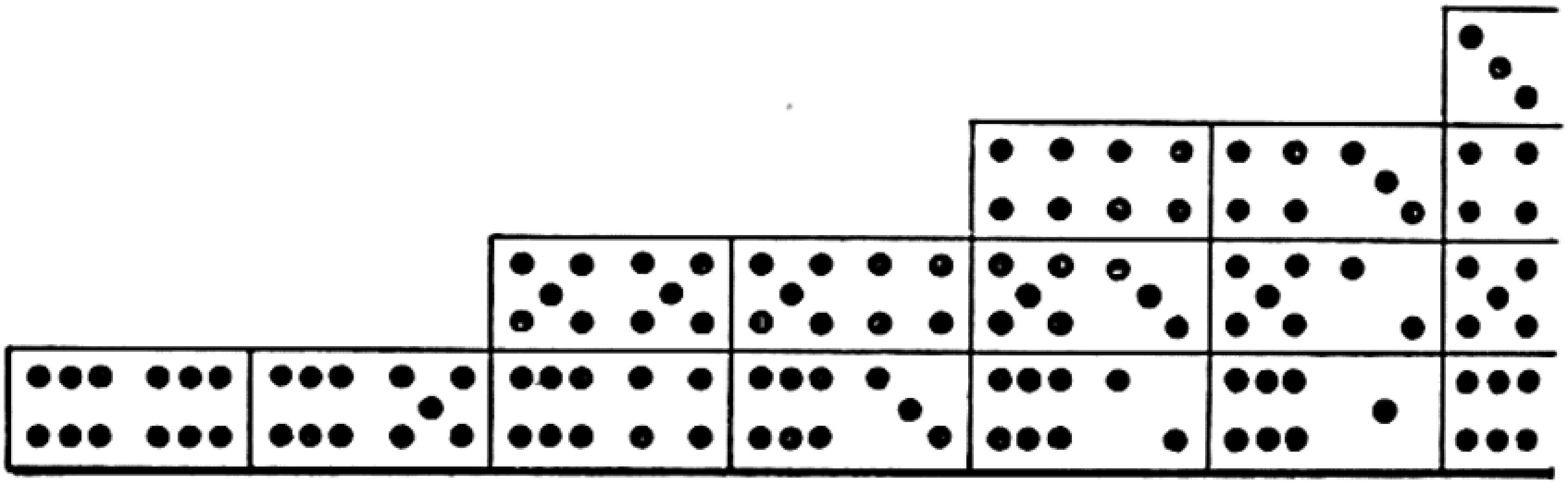
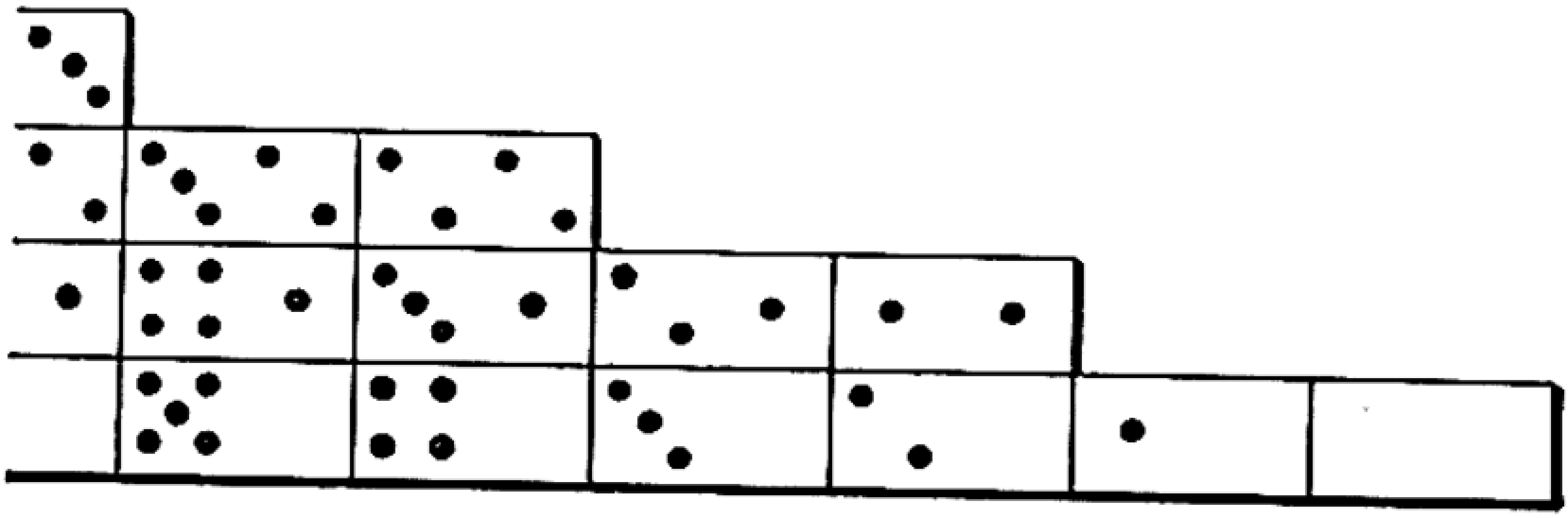


FIG. 26. Domino

'four-er' and a 'three-er' were put together to form a 'seven-er'. Situations with this property of two or more clearly defined groups also offer a choice of group as the first to be counted. It cannot be assumed that a child understands number behaviour well enough to know that it does not matter which group is counted first. He only learns this from experience.

A child can only dispense with counting if he has had previous experience of the numbers involved, has found the result memorable, and realizes that a count of pennies will be, for example, the same as a count of marbles or flower heads. The example chosen of groups of 4 and 3 is not one which is likely to be memorized early. Four and three are not nearly so common in experience as a pair of fours or threes, or successive groups of two. But Rosemary knew before she went to school, without anything but normal experience, the results of combining a good many different small groups, or rather, if she had to deal with a situation involving two groups she could combine them without making a count, though she might not have answered the question 'What is 5 and 3?' She had learned to cope both with a change of context and with a change of numbers within any context over quite a wide range. Teaching sometimes fails because it concentrates on the second of these, that is, varying the numbers, in some limited context of, say, adding groups of counters. Rosemary was worried by being expected to do little sums of the type $4+3 = 7$; she did



stair, No. 2.

not really need them as an exercise in reckoning, and she did not know what they were all about. After a while she adapted herself to the trick of moving counters and ceased to be worried, though the work had no significance for her and, if anything, interfered with the good habits which she had previously acquired of taking note of numbers. The work had no relation to her interests and was too featureless to be memorable. She had, in fact, made acquaintance with an inert idea.

The word 'and-ing' has been coined for the title of this section to emphasize the stage of experience which precedes the recording of the number content of situations leading to addition. During this stage the putting down of figures as a memorandum might occur spontaneously or in response to advice, for example, in shopping; but widening of experience at this stage is more appropriate than concentration on number content.

Recognition of groups on dominoes has been referred to in Chapter I. An arrangement of a full set which involves the recognition of the sum of the groups is given in Fig. 26. It might well attract children at some stage, especially those who had used the stair of Fig. 9 to check the set of dominoes. Only the central column, 6, and those to the right of it give a full analysis of the totals, since the dominoes limit the separate groups to 6 and under. Thus seven and five to make twelve is not included in this pattern.

Fig. 26 is a good example of an occasion for totalling groups without writing down any little sums such as 6+6 = 12. While numbers are small at all events, 'plus' can wait. Although it is very seldom in later stages that some other sign is used where plus is the appropriate one, the converse is not true; 'complementary addition', mentioned below, is one cause of confusion. Even the word 'and' which is so nearly the live counterpart of 'plus' has its snares. In the Nigger Boys, for instance, 'and then there were six' refers to cumulative circumstances and not to the numbers involved.

Sheila, who said of her own and her sister's ages, 'I'm 4 and Celia's 7. When I'm 5, Celia's 8; when I'm 6, Celia's 9. She is 3 years older than I am', recognized a situation the number content of which might be written, 9−6 = 3. It is, however, normal to reach such simple reckoning by 'complementary addition', that is, counting on from 6 to 9, or knowing that 6 and 3 make 9; 4 and 3 make 7, 5 and 3 make 8. This was a memorable occasion for Sheila which might well fix in her mind not only the (6, 3, 9) relationship, to which she was probably already attuned, but also the (4, 3, 7) relation and probably the (5, 3, 8) also.

Situations involving comparison by difference are probably as common in a child's experience as those which lead to finding a total, and they are generally more interesting. At this counting stage, they are worked out by complementary addition. Fig. 13, Chapter III, shows a very early interest in difference of height, the caption for which is 'Alex will have to grow 6 inches. . . .' Scoring games are often introduced about this stage, but their interest for young children is rather overrated. Their value even as a numerical exercise depends on playing the game as a game, so that the score may be urgently interesting as a measure either of skill or of success; in the first the scores made by a single individual may be compared to see if they improve, and in the second different individuals compare their scores which may involve a total of several turns. Fig. 27*a* gives Tom's separate supposed scores set out by counters to show vividly a trend of

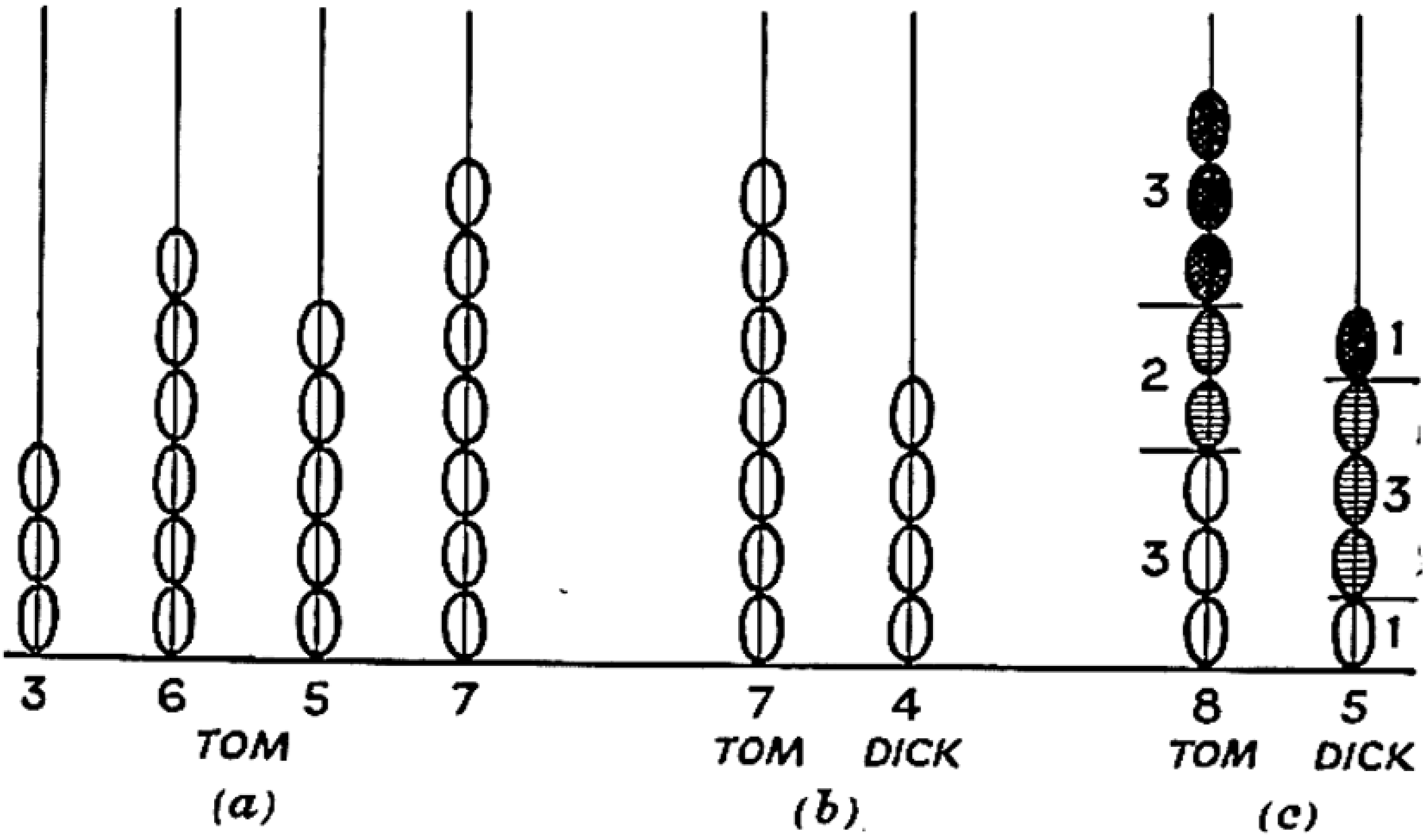


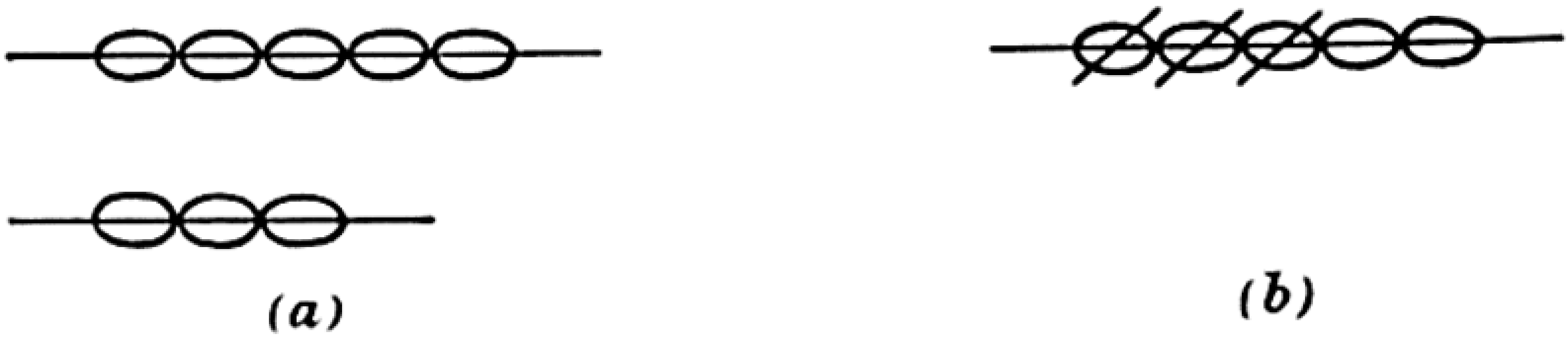
FIG. 27. Totalling and comparing scores.

improvement with practice: Figs. 27*b* and 27*c* show comparison of Tom's and Dick's scores, the scores in Fig. 27*c* being made up of the total of 3 turns.

Comparisons are generally of greater interest than totals. Indeed, totals often derive such interest as they have from an implied comparison. The collector's 'I've 20 conkers' implies 'Beat that if you can'. Height or age suggest 'I'm a taller, or older, girl now', or 'I'm taller or older than you are'. 'I've found a six-er' applied to flower heads implies that this head compares favourably with others. An example arising from an activity occurred in a class of older infants.

A group of older infants were jumping in the playground and measuring and comparing their long-jumps. The differences were both measured and counted up and the jumps were set down in feet and inches, because they were measuring with a yard-stick so marked. They used complementary addition as a matter of common sense and when a pair of jumps were, for instance, 3 ft. 2 in. and 2 ft. 5 in., they built up the smaller jump to 3 ft. Soon one child remarked 'We need to know all about 12 to play this game!' They were advised to record and illustrate a ready reckoner for the parts of 12, which they then made use of. Later they found this reckoner useful for shopping, and Alan remarked: 'Twelve is an important number to know. There are 12 inches in a foot, 12 pennies in a shilling, 12 in a dozen, 12 months in a year, and 12 hours on the clock.'

Shopping situations in which change is given form a bridge from situations involving comparison to those in which taking away occurs. Although the number relations involved are the same and the reckoning for small numbers is usually obtained by counting on from the smaller known number to the larger, that is, by complementary addition, comparison and taking away are situations of different type. In comparison the two elements have a separate existence (Fig. 28*a*), whereas in taking away (Fig. 28*b*) the smaller known number is a part of the larger. In (*a*) we have the type of comparison of jumps, ages, scores, &c. In (*b*) we have the type for eating sweets, losing marbles, spilling milk, using up, giving away, &c. The important example of giving change involves the partition of the money you have into parts, one part to pay and one to keep, and is similar to (*b*) if treated only from the purchasers' point of view.



$5-3=2$
Result of comparison,
2 more.

$5-3=2$
Result of taking away,
2 left.

FIG. 28.

As with comparison, taking away seems a more vivid experience than totalling. For instance, if shopping involves buying a 5*d*. doll and a 3*d*. ball, the separate prices are lost in the total 8*d*.; but if the shopkeeper then gives 4*d*. change from 1*s*. the 4*d*. has the aura of money still to be spent and the (8, 4, 12) relation is more memorable than the (5, 3, 8). The exercises in number composition which were more common formerly than now are of this 'giving change' pattern. They have some advantage over featureless little additions, but they really belong to a stage of systematization rather than to a stage of experience. The children described above who wanted to know all about twelve had reached this stage of systematization for this number. They also brought together into one relationship the two types—comparison of jumps and change from a shilling in shopping. They also no doubt used their table of parts of 12 for finding totals when appropriate. The record was actually made in its addition form.

The two types of situation giving rise to subtraction and the complementary nature of subtraction and addition may be lost in confusion if reckoning is made to take the place of experiences and experiments covering this field. The symbols plus and minus belong to the stage of accepted number relationships arising from a wealth of experience and to the late stage of appreciating reversible operations. A few of the examples given suggest that one means of filling out experience is to include simple measures within the field of and-ing, comparing, and taking away, before these are reduced to formal number statements. Once counting measures has become an accomplishment, there is nothing to prevent this. The children who compared their jumps went on to compare the ages reckoned in months of the babies at home, and they used familiar measures of capacity, the quart, pint, and school milk bottle, in a great variety of ways.

A most unfortunate procedure is to limit the child's practical purposes to match his supposed progress through arithmetical processes. The experience of finding a way out by dealing by means of counting with the objects themselves (or their counterparts, e.g. sticks) prepares the way over a broad front, and even within arithmetic the idea that *A* necessarily comes before *B* may be quite ill founded. Some of the sequences in arithmetical syllabuses are as conventional as the alphabet itself.

Situations involving subtraction may involve difficulties which, though amenable to solution by first principles, are at a much later stage overcome by more general statements involving 'signed' or 'directed' numbers. For instance, using first principles, if *A* owes *B* 8 pence and *B* owes *A* 5 pence, the word 'owe' immediately suggests a taking away of money from assets. The situation is, however, a matter of comparison. The difference between *A*'s debt to *B* and *B*'s debt to *A* is the difference between 8 pence and 5 pence in favour of the larger debt, i.e. effectively *A* owes *B* $(8-5)$ pence. Stated from *A*'s point of view, using minus for debts and plus for assets, *A*'s change of balance after the transaction is (-8) and $(+5)$, i.e. (-3), whereas *B*'s is $(+8)$ and (-5), i.e. $(+3)$. This statement ignores the comparison and keeps to one point of view at a time. It is the better method of dealing with a system of debts and payments in which more than two people are involved.

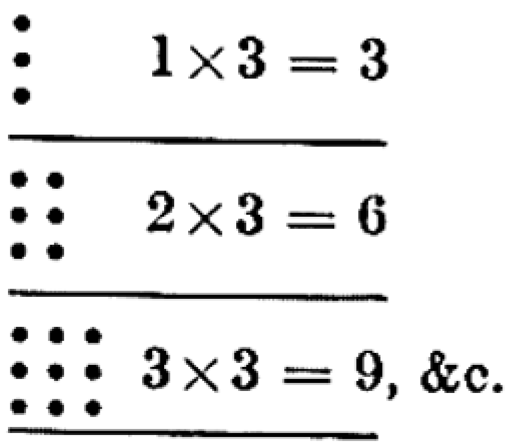
Another kind of example involving comparison and a change of viewpoint has been given in the story of the two children arguing about which Victoria penny was the older.

It is adjustment to such situations as these which helps to prepare the way for a much later introduction, generally at the secondary stage, of the notation of signed numbers in which numbers may be considered as marked out from some zero (or viewpoint) in opposing senses; in the two examples quoted these senses indicate assets or debits and years after or before a zero date. The tale of the Victoria penny also involves the idea of a change of the zero from the Christian Birthday to now. For the earlier years of the primary school the point of interest for the teacher is that any particular situation which is most appropriately expressed in this way is likely to be a more difficult one to disentangle by primitive means. Like the Victoria penny, however, these situations do arise, though they may not be sought after so early.

(*b*) EQUAL GROUPINGS

Equal groupings occur frequently in experience, and counting group by group instead of one by one can be encouraged from quite an early age. Nevertheless it is not infrequent to find multiplication tables taught as a detached feat of memory, together with only a little explanatory matter with counters and diagrams. This section is called 'equal groupings' in order to emphasize stages of experience which may well precede the introduction of multiplication and division as such. An example of lack of variety in preliminary experience and of premature formal teaching follows.

> Royston was one of the oldest and ablest children in an infants' school in which the teaching of 'multiplication tables' began at any early age, accompanied by charts of the type
>
> • • • $1\times3 = 3$
>
> • • • $2\times3 = 6$
>
> • • • $3\times3 = 9$, &c.
>
> Even this chart is open to misunderstanding in comparison with



> but Royston had no difficulties arising from the chart. Effective teaching had indeed made him very skilful in dealing with numbers. He knew a lot about numbers between 20 and 30. He knew that 28 came in the table of fours and that it was 7

> fours. He knew that 24 was 2 twelves, 3 eights or 4 sixes. He picked out 23 and 29 as prime numbers though he did not call them that, and when he included 26 as another number not coming in a table, he was greatly amused at finding it in the table of two's when he carried this on beyond 12 twos. Royston, however, had very little general knowledge about the uses of small number in his everyday life, and he was much disturbed to find other children in his class at least as knowledgeable as himself in answering questions about the milk bottles, the calendar, and so on. Presently he asserted himself by remarking 'I know how many farthings there are in sixpence!' When questioned about this he said, 'I did it at home. I wrote it down on paper and added it up'. It was suggested that there was a shorter way, to which he replied: 'Yes, I could add up threepence and then I could put down another 12 and add that on.'

Had the suggestion been in the form of the question 'What did you write down on paper?' Royston might have said 'Four and four and four . . .' and have then been encouraged to use the familiar form of words 'Six fours' which would have linked his experience with his almost meaningless tables skill. This able boy would, no doubt, fill out his table patter with meaning and usefulness in later years, but examples are given later of children who fail to do this throughout primary school life, and even Royston almost certainly will have lost in time, initiative, and more appropriate learning outside the field of reckoning through an early and narrow concentration on number accomplishment. Less able children would be at greater disadvantage.

Some types of experience, which involve equal groupings and which, therefore, lead up to understanding of multiplication and division, have been referred to in earlier chapters. They occur from an early age and do not need to wait for any formal treatment of addition and subtraction. Counting in twos, for instance, is a rhythmic short cut which follows easily as a sound pattern from

One, *two*, button my shoe
Three, *four*, shut the door, &c.

The sound pattern takes useful shape as a means of totalling the many things which go two by two, children in twos in desks, handles on a chest of drawers, eyes, ears, and so on.

Again, counting in tens arises from our counting names associated with counting fingers, and is a matter of number interest from the observation of written numbers in the environment, as has been developed in Chapter II. In this chapter, also, stress has been laid on the symbolization of a recurrent group, whether of tens by its position in written numbers or of tens or some other number by some special token. The example of a two-shilling piece standing for two separate shillings is given as an instance arising in experience.

The stage of counting in groups in relation to experiences calling for particular groupings begins early and lasts long. In a real situation, if the number of groups involved is not immediately obvious, group counting is often preferred to multiplication even in adult life; for instance, to find the number of items, such as cups arranged in an orderly 5 rows, it may be as quick to count 5, 10, 15, &c. to 60 as to count the 12 in a row and say 5 twelves are sixty. The order in which skill is acquired in group counting is not an order of magnitude of group, the count of threes, for example, is rather hard; the difficulty is partly a matter of experience and partly one of number pattern in relation to counting in ones and tens. The following notes on the counting of small groups other than 2 and 10 are given in order of group magnitude, not difficulty.

The acceptance in measuring of units of related size offers one of the most important fields of experience of equal grouping. A yard-stick with 3 feet marked clearly on it, if used as a measurer of a long length, gives in practice either a count of ones if the unit chosen is a yard or a count of threes if the unit chosen is a foot. Again with the 3 school milk bottles which make 1 pint, the count in pints may be associated with the count in school bottles. In this the presence of pint bottles would be required to make the experience equivalent to that of the yard-stick. In both these experiences a count of threes is involved; ‘Three, six’ is quite an early accomplishment, but ‘Three, six, nine’ marks a significant stage in number familiarity, and little children often say eight instead of nine or count the third three in ones.

‘Four, eight, twelve’ has associations on the one hand with the count of twos and on the other with common experiences of groupings in fours, 4 legs, 4 wheels, 4 sides to the table, and so on.

Counts of fives have a pattern easily recognized in number naming. In experience groups of five occur in the grouping of minutes on the clock and in the familiar 30-bottle milk crate.

Counts of six also occur in experience of the 30-bottle crate and can be associated with the common count of twelve on the one hand and with the count of three on the other.

The seven grouping is a difficult and isolated one. Days in the week offer an early illustration, but the most useful accomplishment here is to find out the date next Monday (or other day) just a week hence, and this involves not counting in sevens but counting on seven from any given number. A chance example of counting in sevens is quoted in Chapter III. A boy whose foot was 7 inches long used his foot to measure a length involving a count of fourteen sevens. Such an activity might give a child a proprietary right in a particular count, in this example seven.

18 Eights and nines are associated with the counts of twos
27 and threes respectively. The count of nines if written in a
36 column shows a written pattern almost as appealing as the
45 count of fives. Children often notice the‘ One up and one
54 down’ of the tens and units and so gain a pure number
63 experience appropriate to able children at an early age,
72 and attractive to the less able, also, if they are given time
81 to approach tables through experience both real and num-
90 erical. The ‘ten and one less’ approach to counting in nines

is the kind of attack which is best left at this early stage to the children themselves to discover. Such devices offer a happy field of number experiment for children inclined that way. Readiness for short cuts of this kind depends on how far a child is accustomed to number behaviour, in this case that two successive operations can be equivalent to one different one. At the other extreme, however, primitive methods allowed till they become a habit simply encourage lazy-mindedness.

> Jennifer, aged 8½, and a sensible lass, still counted out everything on her fingers. Some remedial tuition encouraged her interest in pairs of numbers and in a general way discouraged counting in ones as rather babyish. Fourteen and fourteen occurred unexpectedly in experience and it was observed that she thought out rather than counted her answer. She explained that she knew twelve and twelve was twenty-four so that thirteen and thirteen was twenty-six and fourteen and fourteen twenty-eight.

This was an extension of what had been noted in remedial tuition.

Counting in twelves is a most useful accomplishment easily associated with piling pennies, sorting dozens, or with inches in a foot.

The practice of counting in groups is a preliminary to the systematic knowledge of how many groups go to a given total; it is associated with the feeling that certain numbers will, and others will not, be reached in particular group counts: for instance, 25 may be recognized as belonging to the count of fives, and 22 discarded from the count of fours which is known to contain 24 because of the number of farthings in sixpence. The feeling for this (4, 6, 24) relationship at this stage hardly favours multiplication more than division.

Furthermore, certain items will stand out clearly from experience. Thirty may be known as 6 fives from 30 minutes in the half-hour and from the count of the milk crate. This establishes a firmly known number result, and also the beginning of a meaningful relation between 6, 5, and 30 which might, in favourable circumstances, be carried over to such a question as How many sixpences in 30 pence ? Such a carry over is unlikely until there has been plenty of experience both of totalling groups and breaking down totals into groups of particular size. On the level of practical experiment this makes no demand on knowledge of number combinations, though it may well contribute to it. Money gives very good experience: for example, four threepences to 12 pence and its complement, How many threepences in 12 pence ? Measures can give experience of the two aspects of the relationship in practical form with quantities instead of numbers of units, thus 5 cupfuls to this jugful has the complement, How many cupfuls to the jugful ? Five. Figs. 16 and 17 in Chapter IV show the two aspects of the same situation, 6 cars with 4 wheels, each examined in the first case by totalling wheels car by car and in the second by assigning groups of 4 wheels until all 24 have been fitted.

In the above experiences the emphasis is on the size of the group rather than the number of groups. Sharing situations on the other hand make the number of groups the dominant factor. The shares receive first one apiece, then two apiece, and so on until all are exhausted. The difference of aspect shows up clearly in the sharing of a jugful of liquid. Supposing enough vessels exactly alike to be available for the shares, then the result of sharing is a quantity which is the same in each vessel when the jug is emptied. The sharing into 5 parts is shown diagrammatically in Fig. 29. The quantity in the cup has to be taken 5 times to

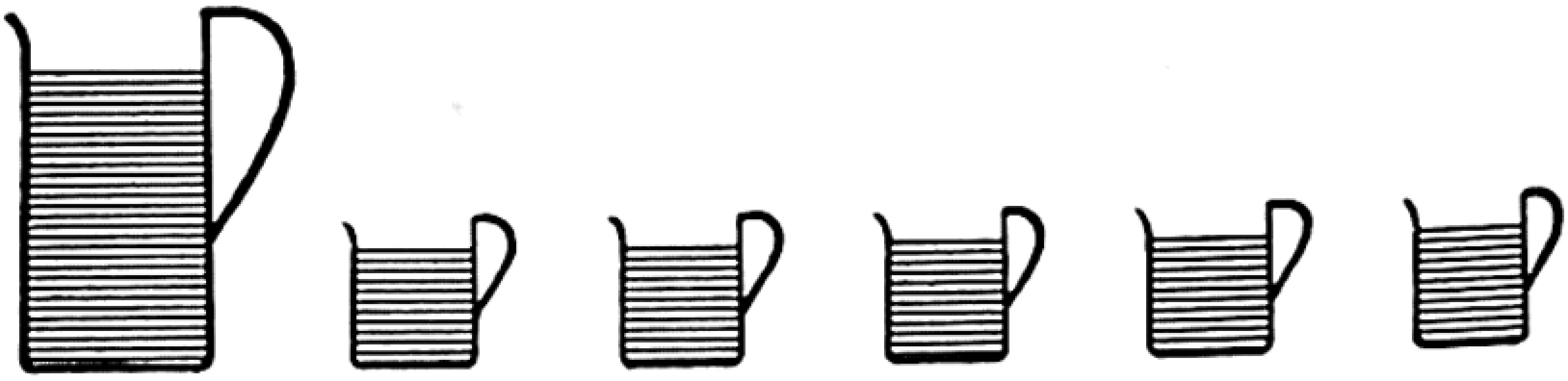


Fig. 29.

make up the jugful. Similarly, if 15 sweets are shared among 5 children each gets 3 sweets and 3 sweets taken 5 times makes a total of 15. The form of words twice, three times, five times, &c., puts emphasis on the number of repetitions rather than on the size of the group repeated. Situations involving increase in size lend themselves to this form of speech, for example, 'draw it three times as long'. To say 'you have 8 marbles and I have 16, I have twice as many', shows a clear appreciation of a (2, 8, 16) relationship that differs from '2 eights are 16' with its suggestion of repeated addition. There is a wide variety of experience to be covered before breaking down into equal groups and building up from equal groups becomes familiar. To keep numbers small helps familiarity by increasing recurrence of particular groups of numbers until, for example, the (3, 4, 12) relationship becomes known in its various aspects. The word comparing, which has been used earlier in the chapter for comparing by difference, might have been introduced also for comparing by quotient. Early intuitions about comparative size are very precious and blunting of sensitivity by early mechanization of a process like division may well account for some of the difficulties which often appear later on in dealing with comparison by ratio. This is dealt with again in Chapter X.

Another important characteristic of number behaviour may well become familiar at a practical level before any formal approach is made. The interchangeability of factors should have its roots in experience rather than in numerical results. Thus, in the 30-bottle milk crate we can examine the number of bottles by counting 6 rows of five or 5 rows of six.

> Patricia, aged 7, was asked how many stitches she had knitted. She had 8 stitches on a row and had done 10 rows. Being a practical young person she found some small objects to represent her stitches and set them up in rows of 8. Counting in eights was beyond her, so she began to count in ones. Her delight was great when she discovered how much easier it was

> to count them the other way in tens and she quickly reached her total of 80.
>
> Mary, aged 7½, wanted to cover her dolls' house bathroom floor with 'tiles'. She found that it needed 8 of the coloured cardboard pieces for a long side and 4 for a short side. After some consideration she asked for 32 tiles, explaining that she could count up 4 rows of eight or 8 rows of four.

Mary could write this: $4 \times 8 = 32$ or $8 \times 4 = 32$. Patricia had not yet learned any symbolic expression, but was approaching readiness for it.

It has been the intention of this chapter to suggest that a great variety of experience may precede formal expression of its number content or systematic knowledge of number bonds. Important ideas such as the complementary nature of addition-subtraction and multiplication-division and the interchangeability of factors in a product can be rooted in experience and a practical flexibility. The idea of reversible operations matures slowly. On the practical level of solution it does not matter if situations are taken out of turn. Folding a length of ribbon by trial and error into a given number of equal pieces or sharing out sweets one by one is an experience of sharing which requires little number technique. For example, sharing sweets into 3 groups can be followed by a rhythmic count either by groups, if that is possible, or in ones with an accent at the group end thus, 'One, two, *three*; four, five, *six*; seven, eight, *nine*; ten, eleven, *twelve*'. The beginning of a connexion between three four times and sharing is established.

Although knowledge of particular number bonds will grow by association with memorable or recurrent practical use, the only systematic advance envisaged at this stage is that from counting in ones to some instances of counting in equal groups.

The forms of expression appropriate are those of normal speech. The use of the word 'times' is suspect, except where the context clearly demands it.

Teachers who appreciate the possible scope of experience will find relevant opportunities in the handwork and general life of a class. The use of money and measures may often provide a starting-point. The care and distribution of classroom materials, collections and inventories, records of achievement, and so on, may all contribute opportunities which have an element of spontaneity. The desire to exercise increasing skill in group counting may from time to time make the more artificial approaches to circumstances which involve equal groupings acceptable, that is, acceptable in the sense that they are undertaken in an adventurous and experimental spirit.

One final example will throw this pre-tables experience into relation with the continued need for similar experience in later stages.

> A group of older infants had done a lot of work connected with the distribution and costing of school milk among their many experiences with money and measures. They were familiar with the expression one-third of a pint in connexion with the idea of 3 bottles of school milk making up 1 pint bottle. They knew all about their 30-bottle milk crate, including the fact that it contained 10 groups of 3 bottles and therefore 10 pints of milk. They were told that schools in a neighbouring town used larger crates with 7 bottles on one side and 5 on the other. They immediately said 35 bottles, no doubt adding another 5 to the known 30. When asked 'How much milk ?' Alan said, 'Eleven pints and two over'; 'Two what ?' 'Two bottles'; 'How much milk ?' 'Two-thirds of a pint.'

There are two points here worth noting: the 11 pints was no doubt 10 pints (from 30 bottles) and another pint (from 3 bottles), with 2 over. The two-thirds of a pint was 2 thirds-of-a-pint rather than $\frac{2}{3}$ pint. At this stage the third-of-a-pint was a sub-measure rather than a fraction of a pint. Indeed, the difficulties attendant on remainders are only overcome at this stage in relation to particular cases, and no general completeness attends division until much later on.

> There was an unhappy sequel to this story of the milk bottles. During the next year this able class pushed on with tables and written arithmetic. Its only practical work was, if anything, below the standard of the infant work, and there was little of it. At the end of the year the class was doing short division examples which actually included 35÷3. They were asked again the selfsame question about the number of pints in 35 school milk bottles, which they could not now solve.

They had lost touch with their practical methods based on the 30-bottle crate and the general method of ÷3 was meaningless figuring.

One of the main impediments in schools to a treatment of the background of arithmetical processes before embarking on the teaching of the written processes themselves is the isolation of arithmetic from the rest of school life. For most children, until quite late in junior school life, the amount of work of a purely arithmetical character for which they are ready is small.

Even for able children it is seldom that divergence in the nature of the work at early stages is necessary. As a general rule more individual responsibility in wider fields of experience is more likely to meet their needs than greater advance in pure arithmetic, though in fact they will be overcoming difficulties by making more use of previous experience, in the form of greater number knowledge, than their fellows. A word in season may provide a really precocious child with some new means of expression or suggest more economical means of solving his practical problems; but at early stages the able child will be distinguished less by his accomplishments than by his initiative.

VI

NUMBER ACCOMPLISHMENTS: TABLES

> Training implies that work shall become mechanical. Teaching involves preventing mechanicalness from reaching a degree fatal to progress.
>
> MARY EVEREST BOOLE
>
> *Preparation of the Child for Science, 1904*

VIEWED by an adult tables are part of the foundations of reckoning, but viewed by the learner they should be in the nature of a stock-taking of early experiences. If experience is sparse and mainly confined to tricks with counters, tables will be a skill with little meaning or use and the items of the tables a featureless feat of memory, at least until more experience is brought to bear on their use.

The link between tables and a rich experience is a subtle one with two distinct aspects, that of meaning and that of memorable items. Some attempt has been made in the previous chapter to suggest children's growing sense of general number relationships arising during experimental and realistic experiences involving number. There comes a stage, for instance, when situations involving comparison and taking away coalesce into a recognition of differences between numbers.

The use of counters may help children to separate the number content from the particular circumstances under consideration; but there is a double link: firstly the link between the complicated situation and the simple counters and, secondly, between the counters and the number symbols. When the relationship between the numbers is clear and, like the numbers themselves, detached from a variety of happenings, it is time for using a minus or other appropriate sign.

It is difficult to say at what stage mathematical expressions like $12-7 = 5$ and $4\times 6 = 24$ are helpful. In the story of Royston, who knew how many farthings there were in sixpence (Chapter V (*b*)), table learning with counters and symbols was too remote from experience to be available in real contexts. Rosemary (Chapters I and V (*a*)), who knew most of her simple addition bonds up to 12 when occasion demanded before she went to school, was plunged too abruptly into symbolization of them and into a form of expression which applied to numbers rather than to situations. The counters used were for her both meaningless and unnecessary.

It would seem to be a safety measure to attend first to increasing experience over a wide field and to leave symbolization apart from figures till usage demands some compact form of expression. It may well be that the stock-taking we call tables marks this point. The children (Chapter V (*a*)) who, after comparing their jumps measured in feet and inches, decided that they needed to know all about 12 to play this game, were ready to make a compact record of the parts of 12 in the form

$11+1 = 12$ with diagrams at the side
$10+2 = 12$
$9+3 = 12$, &c.

They might also have recorded the complementary form

$12-1 = 11$
$12-2 = 10$
$12-3 = 9$, &c.,

though, in fact, they did not do this. The second form really demands an appreciation of the fact that finding a sum is different from finding a difference by complementary addition, and the difference itself can arise in real situations in two guises, comparison and taking away. None of these refinements would, of course, ever be formulated for or by a young child, but it is well for the teacher to appreciate how much ground experience has to cover if a symbolic statement is to have its full meaning.

The other criterion for readiness for tables is rather easier to deal with. Since tables are in essence a systematization of items of number knowledge, it is well that the learner should first have at least a few such items in his possession. We should hardly expect to systematize a collection of stamps or to arrange flowers in their orders without having a fair number of specimens to classify. Experience provides memorable items to start our number collections. For example, 7 and 5, though a difficult number pair, is much more familiar than 7 and 4 because of its recurrence in such experiences as giving change for a shilling and in the many other circumstances dependent on a 12 grouping. Farthings in sixpence was evidently of interest to Royston (of Chapter V (*b*)) and it might have provided him with a highlight in his table of fours, even though it might not have appealed to his neighbour. Indeed, memorable items are a very personal matter and sometimes include unexpected points of reference outside normal table knowledge.

> Annie, aged 8½, and a candidate for free dinners, solved the problem 'To how many boys can I give 5*d*. from 2*s*. 1*d*?' by noting that school dinners cost 5*d*. and there were 5 school days in the week.

The coveted 2*s*. 1*d*. was of vital interest to her, and she went straight to the point without translating the problem into any general form. This particular problem was tested out at the time in a number of different primary classes and Annie's solution was found to occur about once per class. Annie's neighbour, however, worked with much greater generality and explained his solution with the masterly 'It's how many fives in twenty-five'.

> Rose and Violet, about 7 years old, had become most skilful in using a quoit board, the central and largest number on which was 13. They had three rings for a turn and could score the result of 2 or 3 thirteens as a matter of routine.

Although their result falls outside the conventional table range, 26 and 39 are well within the range of numbers it is convenient to know something about.

Illustrations of number bonds being known because they come in a vivid or familiar context are common in everyone's experience. No number bond is exempt, but some are more likely to occur than others. A great deal of fuss is made about the bonds with figure nought as one member, such as $5+0$. When these noughts occur in experience, in, for example, situations such as the little Nigger Boys, 'And then there were none', they create no difficulty. The difficulties which arise from nought as a place holder in numbers larger than 9 belong to a different category, though it is for the sake of such noughts that these bonds with nought are learned. At the tables stage under consideration it seems a pity to bother with noughts unless they arise in experience, apart, of course, from writing down numbers like ten and twenty.

Addition-Subtraction Table

It has been suggested earlier that in mental reckoning complementary addition links these two tables so closely that they are almost indistinguishable. Circumstances must determine when and whether they are given separate forms of statement. It is not usual or advisable to write all the addition table out systematically at first, though very much later, if remedial work is necessary, items might be summarized in systematic form. There is, however, a general point about addition which is not self-evident to young children. If 7 and 5 make 12—for example, 7*d*. and 5*d*. make a shilling—it is not self-evident that 5*d*. and 7*d*. also make one shilling. This is particularly noticeable in the complementary form requiring the change from a shilling having spent 7*d*. and 5*d*. respectively.

Although it is not usual to make records of the addition-subtraction table, the very common practice of neglecting it because right answers can be obtained by counting in ones is a great pity, especially in some ways of working with counters which confirm this counting habit instead of encouraging an abundance of the bonds known from experience. The making of addition records is not advocated, but neither is an attitude of disregard. Eventually both for mental trimness and for purposes of written computation it is desirable that all combinations of numbers less than 10 up to $9+9 = 18$ should be known without stopping to think. Some of the combinations become true second nature from experience, these include most of the 'doubles' which are often familiar even before counting in groups is far advanced. Infants of normal ability who have a rich experience generally acquire most of the combinations within a total of 12. There are a few more deserving of special attention than others, for instance, $4+3$, $8+3$, $7+4$, $5+3$. These pairs are selected not because of tests, but by reference to results common among children who have had very good experience but still do not know some addition pairs. Children who have confident knowledge of neighbouring pairs will find their own devices to improve on counting in ones such as

$4+4$ is 8 so $4+3$ is 7

$8+4$ is 12 so $8+3$ is 11.

Teaching such devices may be dangerous. If they are overimpressed reliance upon them may continue, thus impeding the carefree reaction of knowing the result. The same is true of the use of patterns, such as the domino, to impress number combinations. These are valuable as part of experience but a possible impediment if used as a habitual step to the result. John, the backward boy of 10 quoted in Chapter I, carried this use of pictures long past the experimental stage until it became a dominant routine.

It is usual to lay some emphasis on number bonds totalling 10. Although these bonds are useful landmarks, the dependence on parts of 10 as a step towards such bonds as $(8+6)$ may turn a result which should eventually be known into a 3 step habit, viz.:

$$\begin{aligned} 8+2 &= 10 \\ 6-2 &= 4 \\ 10+4 &= 14 \end{aligned}$$

The difficult combinations totalling between 10 and 18 are too few to justify discarding counting in ones as the best way of finding out, if the result is not remembered. These combinations are worth examining; doubles may be omitted as generally known; 6+5 is related to double five or double six, similarly all bonds where the items differ only by one can be related to their attendant doubles; 6+8, 8+6 require special attention; 6+9 shares with all combinations containing 9 the property of being one less than the corresponding bond in which 10 takes the place of 9; 6+10 is 16 by an understanding of how we write numbers; 7+4 requires some attention but it is related to 7+5 which, with 8+4 and 9+3, is learned from the common experiences involving 12; 8+3 is similar to 7+4 in its relation to 8+4. 8+5 is also related to 8+4, but is a difficult bond.

Able children by the time they are 7 have often made their own adjustments to all these bonds. Others less able often acquire bad habits by this age due to routine work in adding accomplished by counting in ones. This routine work is sometimes supposed to help children learn the table by giving experience of adding. It may do so in some cases, but children who play for safety by counting in ones, or are too lazy or too slow to do anything else, just become more and more skilful in counting. Experience of using small numbers in their various affairs is likely to make some particular bonds memorable, and when such a basis is available sections of the table can be filled in and practised as a useful tool. If the task of knowing bonds instead of finding them out is not too great, the mastery this gives acts as an incentive. In short, tables are best learned as tables when they are already partly known and when their purpose is clear. At the critical point when finding out turns into knowing, awareness that the result is worth knowing and the opportunity to show at suitable intervals that it is remembered, are helpful to a child; in fact an element of practice contrived either through selected experiences or by oral questioning has its place. This is, however, a highly individual matter as regards both the proper time for learning and the gaps to be filled in the knowledge of number bonds. To question a small chosen group for a short time is a justifiable time-saving device for the teacher, but questioning large groups of children on such matters of fact is almost sure to be extravagant of some children's time. Appeal to both eye and ear is also important in turning what is known into a readily available accomplishment. Speed of response is a different matter; high speeds are not a suitable objective for a stage at which finding out is just giving way to confident knowing, though a very slow response is an indication that a child has had to stop to think or to count. Number games are sometimes used to improve speed of response without strain.

The practice of beginning written addition of tens and units in the accepted form at early ages, while the addition-subtraction table is still insecure, often leads, as has already been described, to perpetuation of habits of counting in ones. Correct results obscure the fact that the table is not only not known but is not being learned. Written arithmetic may provide a field of practice for tables, but it is poor material on which to learn them. It very soon, too, makes its demands not on the basic tables alone but also on what may be called notational extensions of them. In the addition table the extension required is one of very general usefulness which may well be added to general number accomplishments for its own sake. In a continued sum of numbers less than 10 the total may quickly exceed the limits of the fundamental table. For instance, in adding the following scores $4+3+8+7+6$ the partial sums are successively 7, 15, 22; in passing from 15 to 22 the demand is $15+7$ which is an extension of $5+7$ by one ten. This reasoned approach is not the most natural and easy one for children who work more readily from the pattern of a sequence of additions, if such a sequence as the one below is presented to them.

$5+7 = 12$, may be known,
$15+7 = 22$, will at first be found by counting,
$25+7 = 32$, may be counted,
$35+7 = 42$, may again be counted,
$45+7 = 52$, may still be counted,
$55+7 = 62$, may now be found by a recognition of the pattern,
$85+7 = 92$, may be found from the pattern without counting in spite of missing terms in the sequence.

Readiness for such an extension is not solely determined by knowledge of the basic table. It needs the backing of purpose and interest and considerable familiarity with written numbers.

> Herbert, not quite 7, was unusually knowledgeable so far as addition bonds were concerned and generally interested in numbers. An attempt to interest him in extension of the table from a knowledge of $8+6 = 14$ to a series similar to the one above produced no response. His attention was entirely occupied by the number 14 about which he explained laboriously that you had to be careful. You said the 4 first but wrote it after the 1.

He had just discovered the trouble with written numbers that the 'teens' did not conform to the twenties and the thirties in saying tens first. It would have been most unusual to find a child ready for the extended table at this age, although it is unhappily not unusual to find children at this age using counting as their only means of doing formal sums which require extended addition. To use counting occasionally and experimentally in solving some practical problem involving extended addition is another matter.

The corresponding extension for subtraction is described much later. It is useful, but in connexion with division rules rather than subtraction itself.

Multiplication-Division Tables

It is not the details of these tables that should first be known but the sequences of counted groups. In, for example, a table of fours recognizing 28 as a member of the sequence is well on the way to knowing the table. It is, however, probable that general experience may supply some items of the tables as complete items; 3 fours or 4 threes may be known as part of a general knowledge of that important number 12. Other groupings may arise from particular incidents in environment, for example, from window panes, handles on furniture, and so on.

Since counting in groups is an easy and efficient process once it is known, it may well be that the convenience of having a table showing the number of groups counted first becomes apparent in a circumstance calling for division; but a good example in early experience has already been given of a boy (actually rather young for a tables record to be desirable) who used his foot as a measuring rod and wished to record a total of groups of 7 inches. It might have been convenient for him to have been able to look at a table and say, 'Yes, 8 of my feet make 56 inches'.

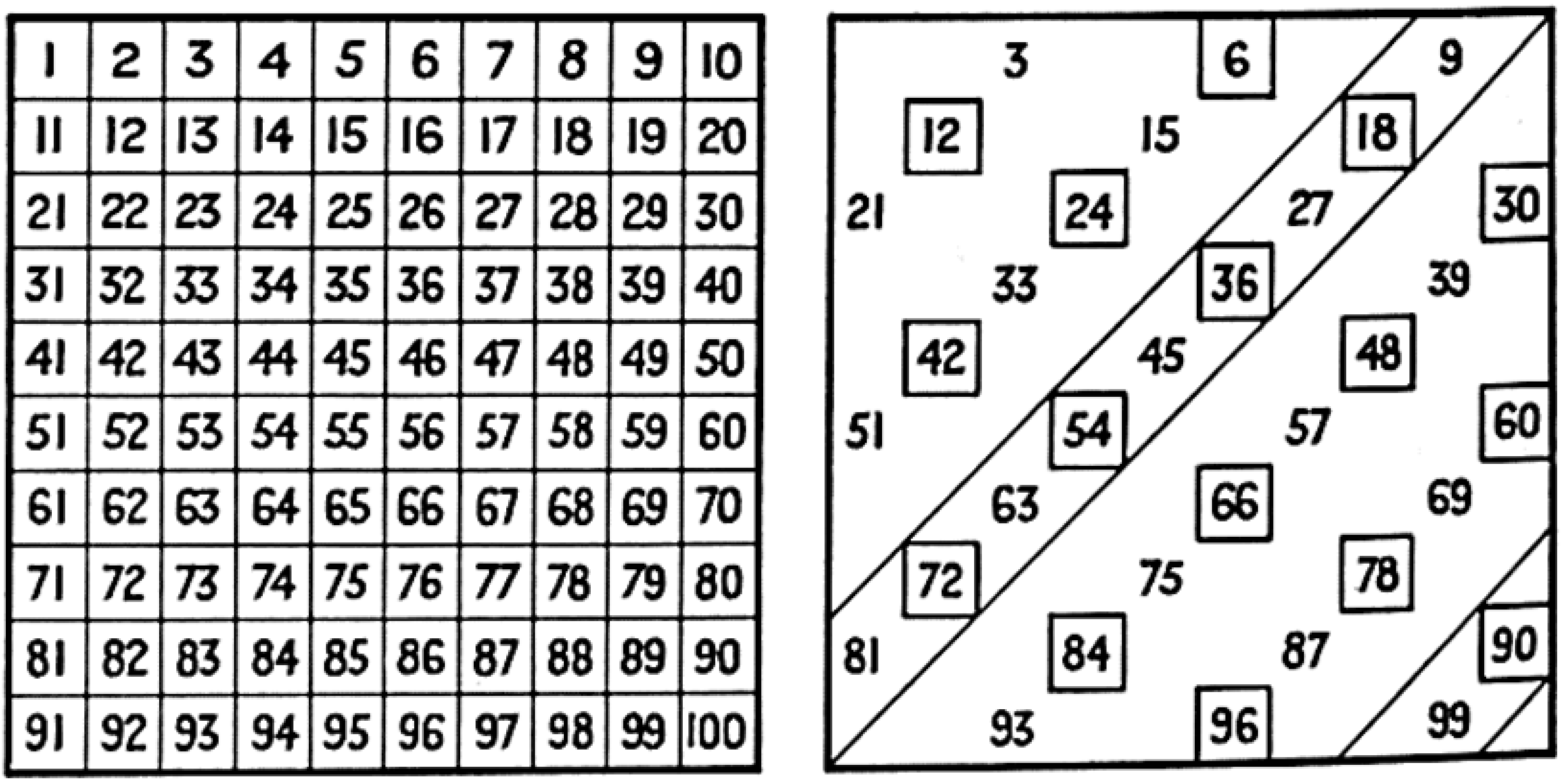
The learning of a table, once it has been recorded, is greatly helped by having friendly items in it connected with groupings that have some real significance, farthings in sixpence 6 fours = 24, 5 minutes in half an hour 6 fives = 30, feet in a 60-inch tape 5 twelves = 60, and the range can be extended to many other examples of money and measures as they become familiar.

Experience of how numbers behave is also relevant. For instance, odd and even numbers are familiar. An examination of the sequences in the table of threes and the table of fours shows:

(1) table of threes 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36.

(2) table of fours 4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48.

In the table of fours, the group counted is even and all numbers in the sequence are even. In the table of threes, however, the sequence is alternately odd and even. It is not beyond children of 9 to conclude that odd numbers occur only when both the group and the number of groups are odd. Thus 7 sixes must be even and 7 sevens odd. Children whose interest in number pattern has been stimulated readily notice such characteristics, though they do not readily give them reasoned statement. Even backward children, for instance, will notice the pattern of the count of nines. Recurrent groups of endings 0, 4, 8 or 2, 6 can be seen in the table of fours, (2) above, according as the number of tens in the number is even or odd. The table of three exhibits three groups of endings 3, 6, 9 or 2, 5, 8, or 1, 4, 7, better appreciated if the table is carried on beyond 36.



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

(*a*) FIG. 30. (*b*)

The pattern which the sequences make in a chart of the numbers from 1 to 100 written in groups of 10 (Fig. 30*a*) is also attractive to children. Fig. 30*b* shows the pattern of threes with the pattern of sixes enclosed in squares and the pattern of nines indicated by the diagonal lines.

It has already been pointed out that experience is the proper basis for the number behaviour of which 6 fives = 5 sixes is an example, or, to put it more fully, 5 groups of six gives the same total as 6 groups of five. Application of this rule of behaviour increases the number of table items known; 30, for example, may be accepted almost from definition as 3 tens, but the number of feet in 10 yards is 10 threes, which is a little less obvious unless $3 \times 10 = 10 \times 3$ is second nature. Again, if 6 fours is known from the farthings in sixpence, so is 4 sixes.

Another contribution, especially to the division aspect of the tables, can be made by building up a sensitiveness to numbers, not by table sequences, but one by one through familiarity with their properties. Thus the numbers between 1 and 10 or 20 are known from many points of view from an early age, for instance 12 can be broken into groups of twos, threes, fours, or sixes, whereas 11 and 13 are unfamiliar and awkward. This sort of knowledge can be extended, as tables knowledge increases, to a recognition of many of the numbers without sub-factors, that is, the primes, and to an interest in some of the numbers which have factors but never come in the tables, which by convention stop at 12 items: for example, 34 is 17 twos or 2 seventeens, and 57 is 19 threes or 3 nineteens. This kind of knowledge of numbers in the table in Fig. 30 *a* up to about 60 is certainly not beyond many children before they leave the primary school.

It may be argued that such considerations have no bearing on children of lesser ability. Yet it is not uncommon to find these children being taught the multiplication table by rote long before it can have useful meaning for them. For the simple minded there are simple counterparts, even of the suggestions made about observation of numbers themselves. Odds and evens over a small range can generally be appreciated. Pattern-making with counters, as in Fig. 31, can be interpreted in terms of groupings or additions. In general, however, it is the pre-tables stage of experience that should be prolonged late into or throughout primary school life for the very slow learner.

For the average learner it is urged that the multiplication-division tables should be approached with purpose and meaning, and their items should be made familiar in as many different ways as can be devised. When tables are significant and familiar there

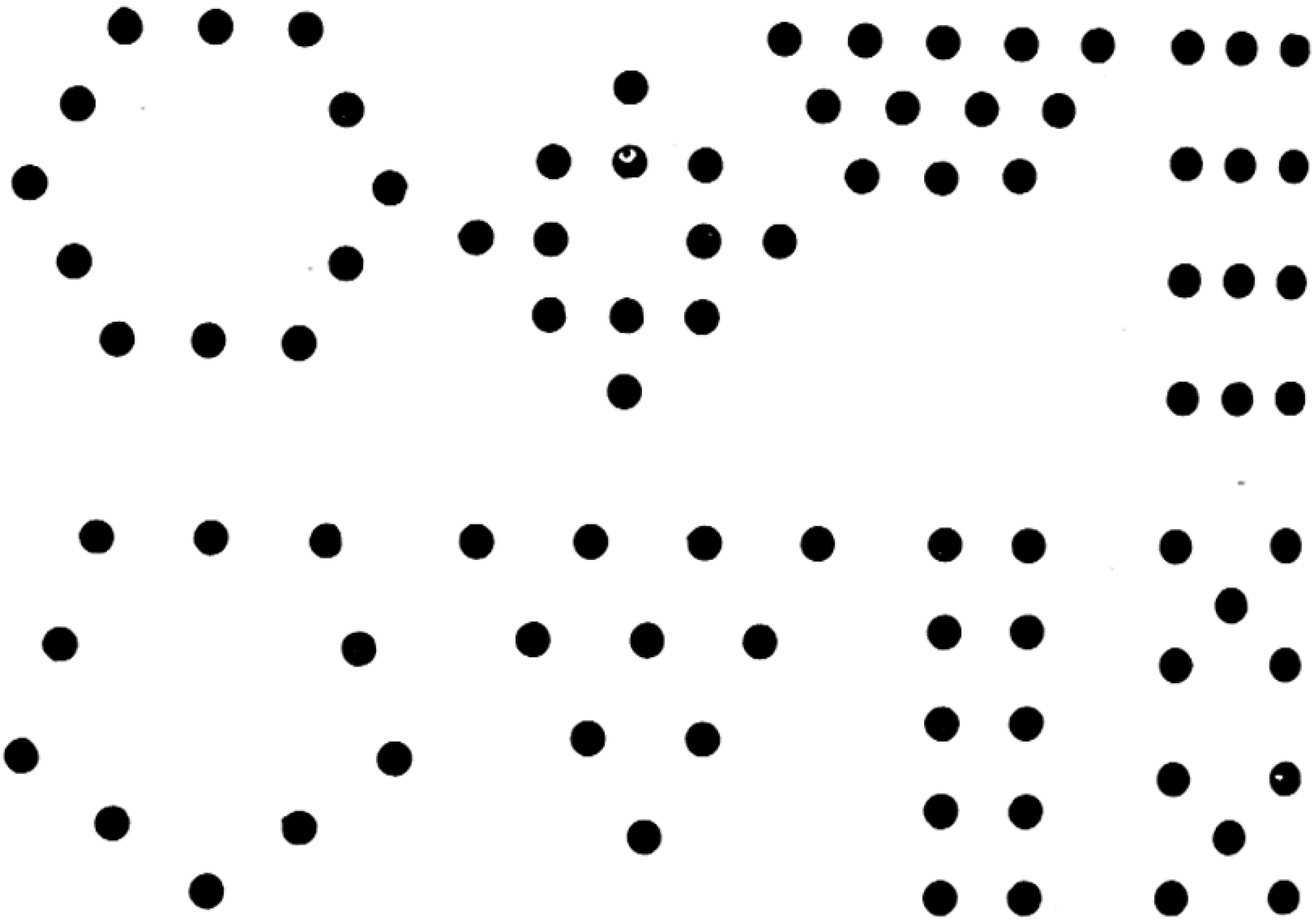


FIG. 31. Patterns with 10 or 12 counters.

still is a place for a routine practice of items to improve acquaintance till it becomes skill. Again, it is worth while to acquire ready response to table items both by sound and by sight; indeed, the sing-song of a table may be part of its pattern provided it is not learned on the level of a nonsense rhyme. There is so much to be done to make multiplication and division meaningful that the routine learning of tables may well be postponed to a later age than is common practice. Nevertheless, the building up of a table and some measure of accomplishment in it does contribute to a grasp of its meaning.

There is for the multiplication-division tables, as for the addition table, an extension of a notational kind which is required for computation with larger numbers, and which greatly adds to the power and range of the tables. This is the ability to extend the known product of two small numbers, for example $6 \times 4 = 24$, to products in which one or both the factors is ten times as great, that is, to 6 × 40, 60 × 4, and 60 × 40. Examples of these extended products occur in changing pounds to shillings; in finding the number of shillings in £8 or £80 we have 8 × 20 or 80 × 20. Again, the number of minutes in 3 hours is 3 × 60. In a later chapter this question is dealt with in more detail; these special cases have to be extended to a general acceptance of products of this kind before rules for multiplication can be treated satisfactorily.

VII

MEANS OF LEARNING: A SUMMARY

FOUNDATIONS of mathematics in the primary school grow like a snowball, concentrically, here a little and there a little. In trying to deal coherently with subject-matter it is therefore necessary to cover not only wide ranges of ability, but wide ranges of age. In the foregoing pages the argument has only occasionally been carried into the normal working of the last two years of primary school life. In what follows this chapter continuity will only call for occasional reference to the infant stages. There should be no abrupt transition between stage and stage in the means and manner of learning and expression, but some gradual change of emphasis. Growing ability to write and to read (to which accomplishments activities which are partly mathematical in purpose may make contribution), widening interests, and personal development have their effect on what children can do and want to do. The very powerful effect of public opinion about what children can do and should be doing at certain ages also has an increasingly strong effect on the children themselves. When this influence is not based on the children's educational needs, it puts great strain not only on children but on their teachers.

So long as reading presents difficulty to a child, his means of learning are limited; he depends on what he does, what he hears, and on the influence of pictures. His means of expression are also mainly action, speech, and picture making. The written word of his own choice may enter the field of expression as early, if not earlier, than does reading as a source of information or direction, because in reading the choice of words is necessarily not his own, but another's. All these early ways of learning and expression remain potent throughout school life, even when reading and writing become easily used tools.

The insidiously simple-looking symbols of mathematics are often introduced early because as mere written forms they are easily mastered substitutes for words. But they are not safe means either of learning or expression. It is difficult for adults to remember that the generality with which they use these symbols is not shared by inexperienced children, who may give to them, and even to words, a surprising degree of particularity of meaning. Particularity in the use of words, however, is generally disclosed by their context. Symbolic statements do not show their possible limitations of meaning, for one of their functions is to omit all trace of context.

Speech as a means of learning and expression includes speech between child and child as well as between child and adult. One of the most difficult problems for the teacher is to create a class environment in which children talk naturally and helpfully about what they are doing, and in which she can herself stimulate progress without the kind of instruction which leads to the artificial spoken responses of a ritual, almost as meaningless as premature symbolic statement. Moreover, critical stages of understanding are essentially individual; the moment of insight may be a solitary one or it may arise in a conversation with another child or with the teacher or in a social community of experiment. It is only in acquiring a top dressing of skill that children of like accomplishment flourish all together under directed group activity. This kind of learning is only appropriate for occasional short periods, especially for young children.

How, then, it may be asked, is a harassed teacher with a large class to deal with so many individuals. Her salvation should lie in the fact that growth is slow and moments of insight few; real progress demands that her children make haste slowly. How she manages her class is, however, a vital factor in mental health and therefore in mathematical progress. Two contrasting class pictures may be drawn, between which lie the majority of infant and young junior classes. Neither is a picture of a particular class, but both pictures are a composition from observed elements.

On the one hand is teacher *A* whose work is judged by the progress her young children make in the forms of mathematics. In periods of 15 to 30 minutes (or longer) assigned to the subject, she gives some oral instruction or questioning to the class or large groups of it, often telling them what they are to do when apparatus or writing materials are given out to them. Work is given out to individuals or groups, and teacher *A* goes round the class hurriedly checking, correcting, and advising. The work may often include wisely graded apparatus, but it is generally beyond the teacher's power to keep all occupied, still less all purposefully employed all the time. Here are three samples from different classes of what may happen to some or to many individuals.

> Terry, aged 5, a lively-minded boy, was content for the first few minutes of a 10- to 15-minute period until he had finished making his figures, but as soon as this was done he was halfway across the table behind him. Before the teacher got round to him he:
>
> 1. Looked at both papers of the children behind him.

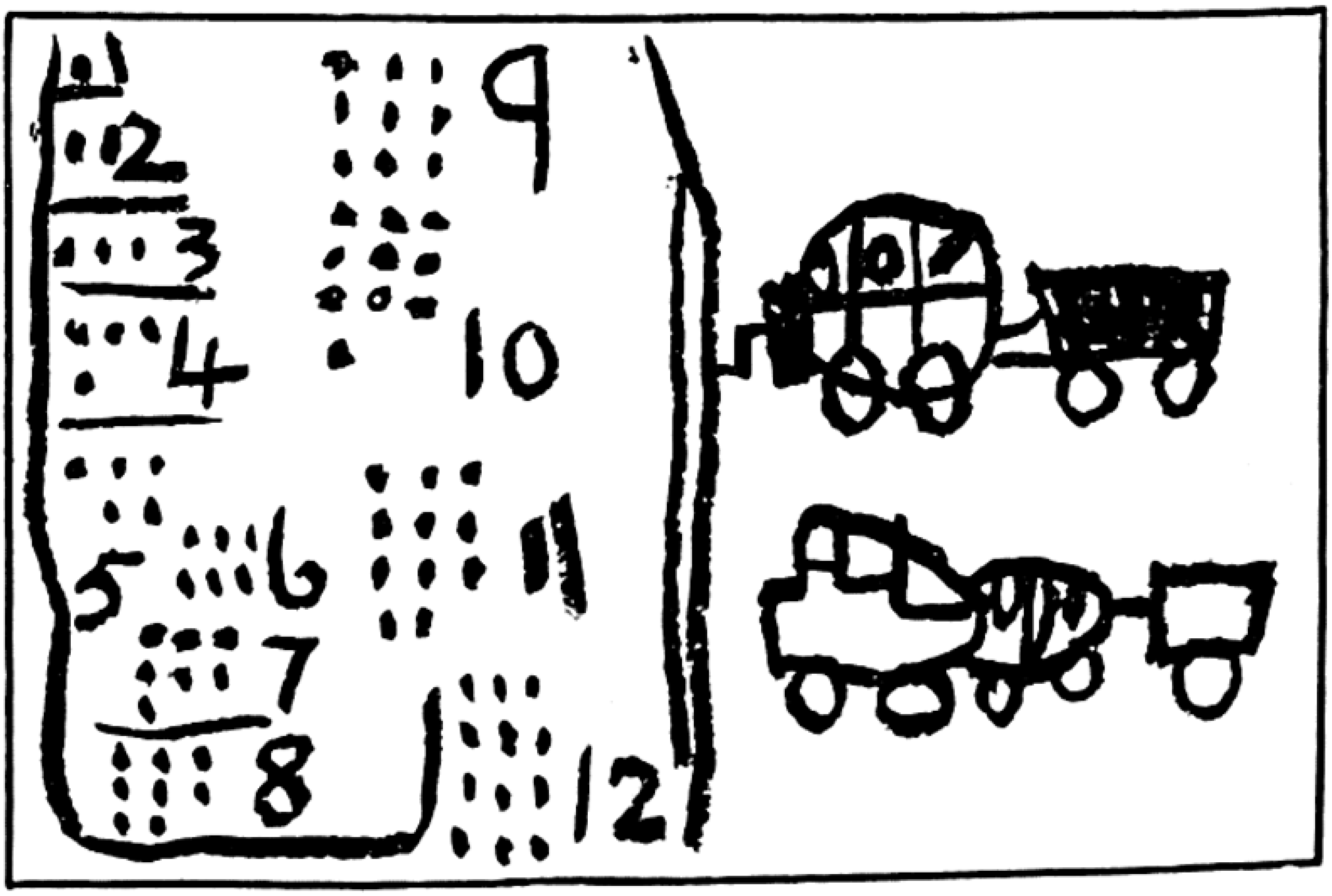


FIG. 32.

2. Picked up his neighbour's paper and said 'Let's draw a lot of cars.'
3. Drew a string of cars (Fig. 32).
4. Banged on the table with his fists and then with the palm of his hand.
5. Tapped his feet.
6. Folded his arms and sat up.
7. Unfolded his arms and examined the soles of his shoes.
8. Wrote on them with a pencil.
9. Got out of his chair, sat on the edge of the table, and swung his legs.
10. Got back into his chair, examined his hands, folded his arms again, and sat up.

Sarah, aged 6, on the other hand, drew a lovely string of pigs not because she had too little to do, but because she had forgotten 'how to take away'. She cried and was excused from the room because she felt sick, and she only returned after the number lesson.

Julia, aged 6, during a 20-minute oral number lesson answered two questions which were asked directly of her (quite a fair share of personal attention in a large class) but volunteered no further information. During the rest of the time she:

1. Removed her hair ribbon and put it in her pocket.
2. Scratched her leg.
3. Sat on one leg.
4. Sucked her finger and drew a wet picture on the desk.
5. Sat on two legs.
6. Examined her cuffs.
7. Returned to her own world, chin on hand, and dreamt.
8. Transferred one foot from seat to ground.
9. Ate a sweet.
10. Took her ribbon from her pocket and attempted to tie up her hair. This lasted to the end of the lesson.

Conversation is one thing, listening to others talking quite another, especially at six when conversation itself is a fairly new accomplishment. How many child-hours go into self-protection of this kind? Adults are given interval music to break too long periods of listening.

The contrasting picture is of teacher *B* whose children are accustomed to deploy themselves with childlike occupation of their choice as soon as they enter the neighbourhood of their classroom. There are a few occasions on which they gather together as a community, and occasionally during such times there may be some community counting, a number jingle, or a note in a news book or calendar. Generally, however, small groups or individuals are busy with a wide variety of affairs, some of which by the uninitiated might be referred to disparagingly as 'play', though a very different play from Terry's and Julia's. There is an air of concentration about what is going on, and movement and speech are quiet and purposeful. It looks simple, but it has involved not only the provision of ample opportunity for active employment, but an understanding amongst the children that teacher *B* does not tolerate the wanderer, the interferer, or the lazy-bones. The demands made by the children themselves on teacher *B*'s attention are not so frequent that they cause confusion; there is plenty to do if she is busy.

> Eleanor, aged 5, stood silently watching her teacher who was completing dinner-money accounts. She nudged away all invitations to join her friends in the dolls' corner (which she loved), saying finally, 'I must watch Miss Browning; that's how you learn.'

Teacher *B* may advise when some group or individual is ready for new work, and she does not hesitate to direct attention to particular jobs to be done during the day or to invite groups or individuals to take their turn in working with her. The materials available may well include many of those given out by teacher *A*, but they are not staled by too constant use, and teacher *B* knows what difficulties and comments they are likely to give rise to.

One of the greatest differences between classes *A* and *B* is that teacher *A* is responsible for most of what is said while teacher *B* holds conversations with individuals or groups of children about what they are doing. Teacher *B* finds almost as many incidental opportunities as she needs at early ages for stimulating interest in the numerical and spacial aspects of experience from the children's imaginative and imitative play, in their constructive efforts, and in the many classroom routines such as noting the time, distributing the milk, counting money, marking calendars and diaries, and so on. She is able to assign duties to individuals in turn, and she knows that they may provide her with the best of opportunities for stimulating interest or carrying competence a stage further. The naturalness of her contacts with the children increases the children's mathematical vocabulary in much the same way as general vocabulary is increased. The children make their own efforts at expression and imitate the teacher's more competent forms of speech as her words become familiar in contexts they understand.

A few examples may be given of different types of opportunity which might arise in any way of working, but which would be quite appropriate and perhaps more likely to be valued by teacher *B* and by her children than by teacher *A*.

A morning routine among some 46 young 5-year-old children consisted of a count of the boys and girls separately. After counting 22 boys, by counting heads as boys resumed their seats, the count was repeated in a chain of large beads from which one had to be removed for an absentee. The chain had 10 yellow, 10 green, and 3 red beads, one of which was removed. The colour changes reinforced the speech rhythm of counting, but no other remark was made during the count. The girls' count followed, also, as it happened, 22. The two bead chains, showing different alternations of colour, were hung up together,

> and a child remarked on the equality. A quiet and almost casual recount was made by the teacher in tens. This might catch the attention of a child whose number sense was advanced for his age. During the morning an occasional child went up to the bead chains to recount them.

This routine opportunity is in contrast to the more opportunist teaching arising from some activity in which mathematical value is entirely incidental.

> Some older infants were planning to use a potato as a tool for producing a repeated pattern. Occasion arose for discussing the unsuitability of the curved surface of the uncut potato for the printing, and the desirability of having a clean cut to work from, so that, before the pattern was cut, it would touch the flat paper everywhere.

Two other examples, the first a spontaneous and the second a prepared opportunity, are also taken from children's handwork.

> Richard, whose diary is quoted in Chapter III, had a great interest in number and unusual skill for his age in counting money. On his own initiative he and two friends measured up 8 yards 2 feet of his peg knitting, with which he had persisted for many days in order to make a coiled mat as a present for his headmistress. The first time he measured his knitting he reckoned 4 feet to 1 yard. He discovered his mistake which he corrected by measuring it again. His formal attainments at the time were few, but foundations were unusually secure.

An example of something introduced to give children a good opportunity for counting, and some association of size with number, is the following pattern which provides, besides good number and plain knitting practice, good repetitive reading, and a useful job most attractive to the children. It was accomplished most successfully by individual older infants.

PATTERN OF GLOVES

Cast on 36 stitches	Knit 7 rows
CAST OFF 14 STITCHES and Cast on 17 stitches	Knit 8 rows
CAST OFF 16 STITCHES and Cast on 17 stitches	Knit 8 rows
CAST OFF 17 STITCHES and Cast on 16 stitches	Knit 8 rows
CAST OFF 26 STITCHES and Cast on 14 stitches	
Turn over for pattern of the thumb	
Knit 16 stitches and turn	and knit back
Knit 18 stitches and turn	and knit back
Knit 20 stitches and turn	and knit back
Knit 22 stitches and turn	and knit back
CAST OFF LOOSELY	

(The instructions were in one column on a folded card and those printed in capitals were shown in coloured script.)

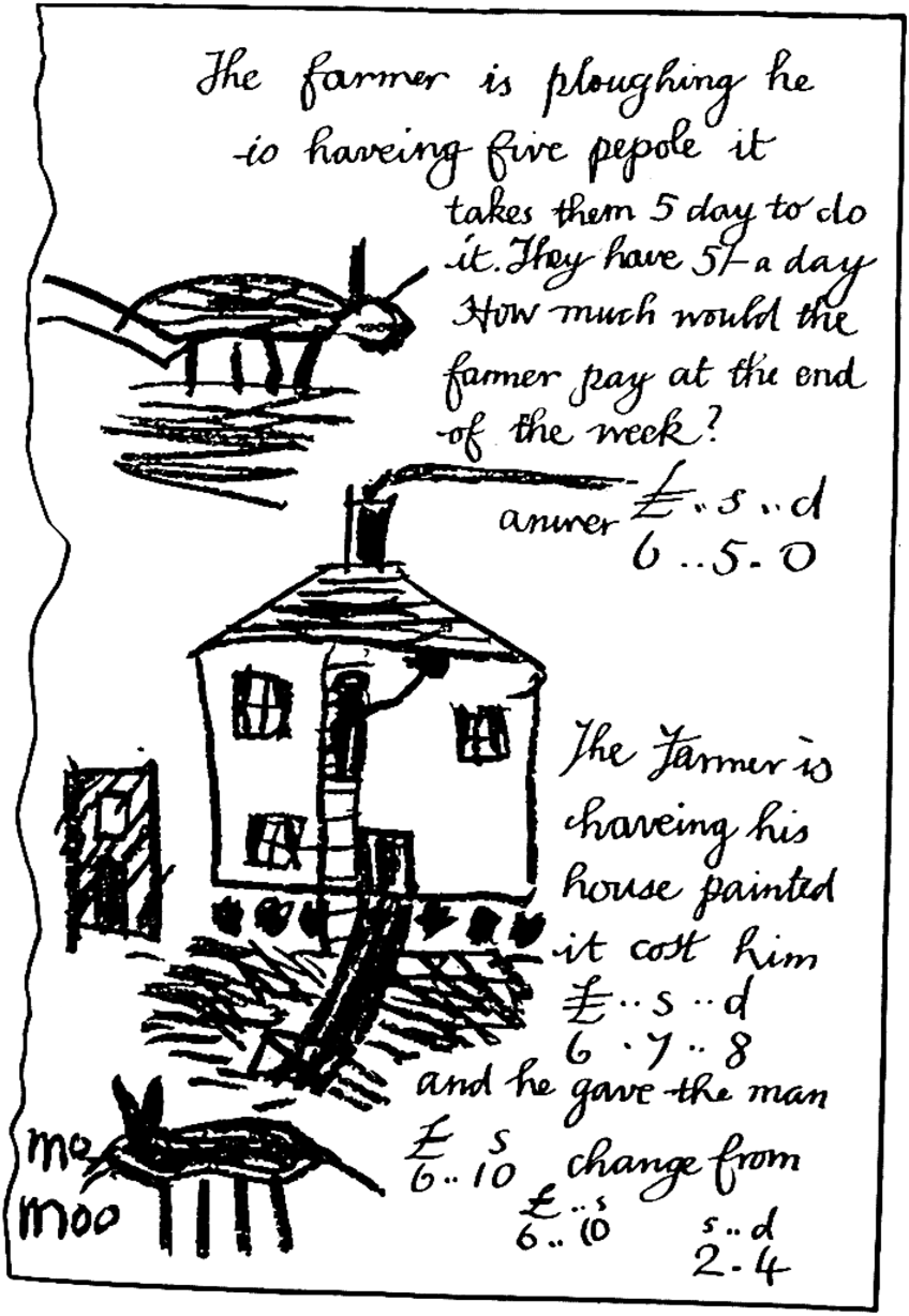
The predominantly oral stage is the most difficult to illustrate, but something of the same principles apply to written records when they become possible and appropriate. If a child makes a record, or talks about something he has done or thought of himself, the words are his. When he answers questions or solves 'problems' presented verbally, the words are to some extent an echo of someone else's. Conventional arithmetical work gives little or no opportunity for the preliminary stage of saying or recording what has been done. It is no wonder that 'problems' and the statement of solutions are so often unsatisfactory.

A girl of 8 years of age and of just above average ability, who lived in a district bordering on a rural area, wove her own choice of calculations into an imaginative effort entitled 'On the Farm'. The items spread over a large sheet of plain paper 15 inches by

20 inches, and were illustrated in chalk so that the whole formed a picture in itself, with the longer side uppermost. Fig. 33 shows the arrangement of the right-hand 6 or 7 inches of the composition which contained two items with gaily coloured illustrations. The remainder was illustrated in black chalk only. It reads as follows:

On The Farm

A lady wants a dozen eggs of the farmer they are 3d each
[picture of egg]
Then she think she sould have some milk 2 qaurts at 10d each
[picture of milk bottles]   s d
3 0
4 ,, 8 said the farmer she gave him 10s   1 8
[picture of ten-shilling note]   4 8
Change From 10s   5s 4d
[picture of two half-crowns, a threepenny bit and a penny]
A Nother farmer came can I have some sheep [picture of sheep]
How many, 10 right you are 1 ,, 16 ,, 7½ each
Have you any separater's yes we have one left
I will have that one   £ s d   [picture of separator]
2 ,, 6 ,, 3   £ ,, s ,, d
Thats all for today   £ ,, s   18 ,, 6 ,, 3
I will give you   20 ,, 16 sir   2 ,, 6 ,, 3
Change from £20 ,, 16s is 3 ,, 6   20 ,, 12 ,, 6
[picture of three shillings and a sixpence]
I am going to the Farm now I am going to buy   s ,, d
3 quarts of milk 10d each   2 ,, 6
2 dozen eggs 3d each   6 ,, 0
2 glasses of cream 4 ,, 4 a pot   8 ,, 8
1 quarter of Butter   1 ,, 6
18 ,, 8
s d
Change from £1   1 ,, 4
The farmer has some cow house's for sale
nobody wants them   [picture of four cow houses]



The farmer is ploughing he
is haveing five pepole it
takes them 5 day to do
it. They have 5/- a day
How much would the
farmer pay at the end
of the week?

anwer £ .. s .. d
6 .. 5 - 0

The Farmer is
haveing his
house painted
it cost him
£ .. s .. d
6 · 7 .. 8
and he gave the man
£ s
6 .. 10 change from
£ .. s
6 .. 10
s .. d
2 - 4

FIG. 33.

The interest of this example, which is only one of many from a small school with rather less than one class per age-group, is that it brings arithmetic within the orbit of free forms of expression. Fig. 34 (p. 99) is a pictorial note from the same school by a child of 8 of less than average ability of things remembered about measures and fractions. The original drawings are brightly coloured.

A very backward girl of 10 years old, who was accustomed to describing what she had done, wrote:

> 'I have made one stride and I measured it. It came to 2 feet. I made 120 strides from one side of the playground to the other. Now I will find out how many yards it is.'

Then follows the necessary multiplication by 2 and division by 3.

Three answers from girls of 8 to a somewhat artificial problem presented verbally may be quoted as typical of well-developed powers of expression. Two of these girls were like the child who wrote about the farm, a little above average in ability, and the other was one of the few really able girls in her class. The girls were asked by their headmistress to explain their answers. They were accustomed to explaining practical problems.

The problem to be solved was:

> 'When Uncle Joe left he gave £5 10*s*. to the family, a pound for Mother, a pound for Father, and ten shillings each for the children. How many children were there?'

The ablest girl wrote:

£	*s.*	*d.*	
5	10	0 —	
2	0	0	for Mother and Father
3	10	0	for the children

£	*s.*	*d.*	
3	0	0	is 6 ten-shilling notes
	10	0	is 1 ten-shilling note
3	10	0	is 7 ten-shilling notes

There are 7 children

The explicitness of the second step is unusual. A second girl dramatized the situation into Mum and Dad, but wrote down no more than was to be expected, thus:

£	*s.*	*d.*	
5	10	0	
2	0	0	for Mum and Dad
3	10	0	for children

7 children

Note also the absence of the minus sign.

The third girl achieved a triumph of statement, for her solution was essentially a practical one and so harder to explain. She wrote

1 pound for Mother
10 shillings for child
10 shillings for child
10 shillings for child
10 shillings for child

1 pound for Father
10 shillings for child
10 shillings for child
10 shillings for child

7 children

The evidence of speech and action is so transient that the argument has been carried forward into the stage when larger numbers are in use together with written expression. The transition to this stage requires further examination, but one more example must be given to illustrate the great difference between a way of working which leads to articulateness with understanding and one which leads to increasing dependence on rules.

Rosie and Billy were 8½ years old and round about average ability. Probably Rosie was of about the same ability as the child whose work is shown in Fig. 34 (p. 99). They were in a school still suffering some of the worst after-effects of war, so that, whereas any one of their difficulties is of recognizable type, so many in so small a compass is happily unusual.

Rosie was given three sums to do but had no idea how to tackle them. She relied entirely on her next-door neighbour for help.

Qu. 1. My slippers cost me 9*s*. 6*d*. and my jumper 2*s*. 6*d*. less. How much was the jumper?

Rosie read the sum, and Billy, who sat beside her, worked it out orally.

Billy. 'If the slippers cost 9*s*. 6*d*., then the jumper was 7*s*. 6*d*., with another 6*d*. off that for the slippers, so that's 7*s*. for the jumper, see!'

Rosie. 'Oh yes, let's put that down' (neither knew how to show the work).

Eventually they wrote down:

s.	*d*.
9	6
7	0

Rosie. 'Is it add up, take away, or multiply?'

Billy. 'Let's look back at yesterday's page.' (They looked back in Billy's book and found that a similarly set sum which they had subtracted had been marked X—wrong—so they decided to 'add up').

Billy. 'I've got the answer, Rosie—15*s*. 6*d*.'

Rosie looked at Billy's book and without hesitation copied down his answer.

Qu. 2. Find the sum of 19, a gross, and half a dozen.

Setting this down presented great difficulty to Rosie. She insisted for a time that 19 should be written 1..9 as they were doing shillings and pence sums. Billy won the day again and explained that it was 'just 19 and not 19 anything'.

Rosie. 'What's half a dozen?'

Billy. 'A gross comes first, and that's 144.' (This Rosie questioned and said she thought it was 124, but Billy was not to be moved and wrote down 144, Rosie doing likewise.)

Rosie. 'Now we write down half a dozen, what's that?'

Billy. 'A dozen is twelve, and half a dozen is 6.'

Rosie. 'No, I think it is just have a dozen; all the lot.'

Billy. 'Of course not,' and he wrote down 6, and Rosie did likewise.

19
144
6

Then followed the same quiet discussion 'add up', 'take away', or 'multiply'? Once more they referred to a previous page and, as before, found a similarly stated sum marked X so they agreed to subtract. (Billy asked permission to be excused, and Rosie was left to her own devices.)

Rosie on request did her working aloud and what follows is a verbatim account of her argument. This part of the story rightly belongs to the discussion of written reckoning in the next chapter, but it is given here for completeness.

19 '9 from 4 I can't put down 1
144 4 from 6 is 2 and one is 3
6 Nothing to take from the 4 and 1 are 5
63 and the one I put is 6.'

The incidence of acute teaching difficulties mercifully, as a rule, moves on from time to time and place to place, but overcrowding settles on some schools for the lifetime of a staff and for many generations of children. The efforts made by teachers to minimize the harm done by persistently bad conditions is inspiring.

Imagine teacher *C* in such a school. Perhaps she teaches the top class of infants back to back with class 2 in a long room. This happily, is a less common circumstance than it was twenty years ago. These two classes and two others are always full to the brim. Dual desks are packed so closely that some gangways are unsuitable for adults. The teacher's space in front is scanty. Room is found here, however, for a children's shop and a pillar-box. Somewhere near the back there is a space for a favourite group model, perhaps a farm; or it may be that this one valuable space contains a nature table or library. On the one easily accessible wall there is a news blackboard on which anyone may write, a news book for special items, and a weather record. Advantage is taken of these few opportunities by having a weekly rota of shopmen and assistants, farmers, and postmen. There are daily orders to be made up at the shop, duties to be performed on the farm, and at least one letter (from the teacher) to be delivered after play. There are also ways in which individuals may use the shop and the farm when they are not occupied by their proprietors for the week, and anyone may write a letter. At their desks the children have most of their day's work in boxes, and though there is only very limited scope for choosing a job, materials for number work are thought-provoking; there is some routine practice work, but an effort has been made to carry the children at least in imagination into the simple number usages of everyday experience. Fortunately, most of the class likes reading, and the vocabulary of shopping cards, and so on, has become familiar. The teacher may say of the shop 'I am so tired of the Rushley Stores', but she knows the children like it, and she perseveres with this favourite routine, only making an occasional change to a fresh type of store. The children may make records like Fig. 34. The social unit in this class is, for reasons of space, either two in a desk or the whole class. Talking to a group is only occasionally possible because it is undesirable to multiply interruptions to class 2.

Nevertheless almost anything, except easy movement, is possible for this class because there is a basic activity of sorts which is well understood by the children and which allows them to enjoy highlights of experience turn by turn. If the contact between child and adult is a sensitive one, there is much compensation for a lack of spontaneity in the background of occupation. In spite of all disadvantages such a class may be at peace, and the children confident individuals making unhurried progress.

The teacher of a junior class has in some ways an easier and in others a more difficult task in trying to create a class background favouring natural rhythms of growth than have teachers *B* and *C* with their infant classes in good and restricted conditions respec-

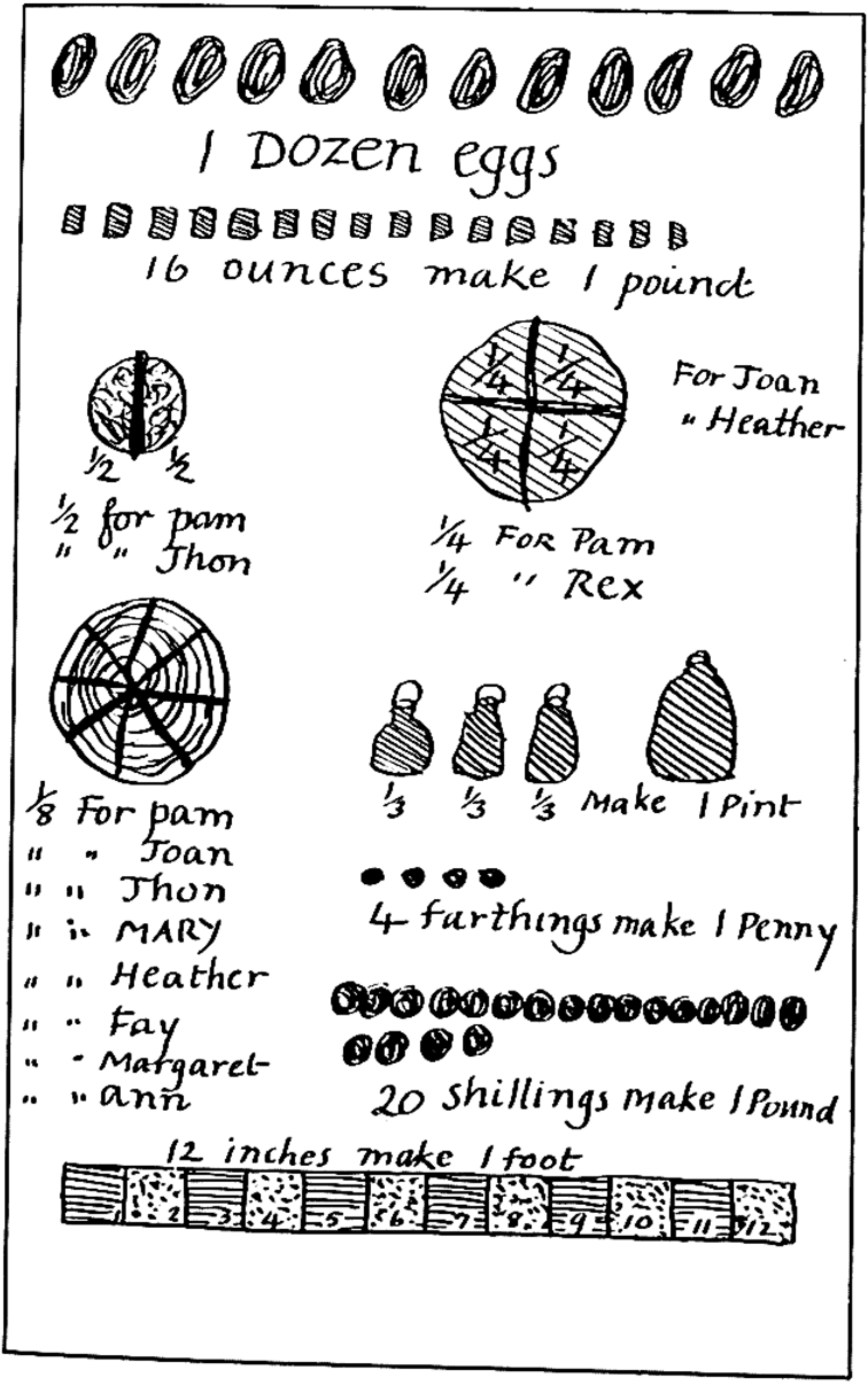


FIG. 34. 'Measures remembered.'

tively. The advantages consist mainly of the increasing use that can be made of reading and writing as instruments of activity and expression, and also, but to a much lesser extent than is common in present practice, of the children's need for pure exercise of their powers over numbers. Against these advantages must be set two main difficulties, one inevitable and the other extraneous. Although children entering the junior stage have in general the common background of having been to school (though perhaps not to schools of similar types) differences in individuals' rates of maturation are showing increasing effects; the gap is widening in the children's readiness for the various stages of learning. Only ways of working which allow for considerable differences can hope to succeed in giving junior children the opportunities to learn from experience that for which each mind is ready. This would not be so great a problem were it not accompanied by the extraneous handicap of popular and even professional expectation that juniors (if not infants) should all, except for a few weaklings, accomplish a set syllabus of written reckoning year by year.

Much skill and energy has been expended in trying to keep up an appearance of satisfying external demands. In the next chapters an analysis is made, in as constructive a spirit as possible, of what is involved in learning written reckoning.

VIII

WRITTEN RECKONING

> I believe that hardly any mistake in education is more disturbing to normal brain-action, more likely to induce nerve-storms in delicate children, or more dangerous to future brain power in all children, than the attempt to convey a new idea by means of a process still artificial (i.e. inadequately co-ordinated) or to teach a new process by means of an idea still unfamiliar.
>
> MARY EVEREST BOOLE
>
> *Preparation of the Child for Science, 1904*

METHODS of computation have been made the subject of a great deal of research and a glut of discussion. For the most part discussion has been concerned with which of several plans for setting down written reckoning is the most successful in practice. Most of the reputable methods of setting down any particular process involve much the same elements in different forms and orders, and any reference to one rather than another does not necessarily imply preference for it.

There has been much less discussion about what makes for an individual's receptivity and readiness for learning these procedures. Any views expressed here, though the outcome of long observation, are not the conclusions of controlled experiment. This long period of observation has been characterized by a steady change of view from an uncritical acceptance of traditional reckoning for young children to the overriding opinion that written reckoning, except in quite small amounts and carefully regulated conditions, is a poor medium of education in the primary school, and any elaboration of it is more appropriate to the secondary stage. Many children of average ability would do better to learn later 'on the job' what is now taught in a detached way at early ages. Abler children, on the other hand, are learning as rules, processes which they would later either supersede by, for example, fractional methods or find out how to do for themselves by common sense. 'Compound multiplication', for example, might as well not be taught in any chosen form. If there were occasional early need for the multiplication of a compound quantity, the process would probably be tackled piecemeal by common sense; it could be tidied up into an economical pattern if repeatedly needed.

Another general opinion is that children learn best the things which serve their own practical purposes. Little evidence except of a negative kind is offered in support of this opinion so far as learning rules of reckoning is concerned, but judging by the kinds of things children do wholeheartedly and by some observation of children when learning about fractions there seems reason to suppose that personal usefulness is an important ingredient of readiness for learning. To state this rather differently, if any procedure in arithmetic, for example written multiplication, is usable for purposes about which the children are themselves concerned, it gains a sanction which acts as a substitute for reasoning of an adult kind. The same might be said about any details which go to make up the procedure, for instance, tables. It has already been suggested in Chapter VI that tables require the sanction of items with real significance, such as 6 fives = 30, from minutes in half an hour, to help make them memorable and usable. Co-operative understanding of procedures of reckoning has to be built up of a number of such sanctions as well as from the overall sanction of the usefulness of the procedure as a whole. The word sanction is used to indicate something acceptable in experience. Some able children seem to arrive at these sanctions in imagination as well as through actual doing, but it is unsafe to trust to verbal evidence; a verbal short cut is sometimes misleading.

Each procedure in turn should, therefore, be approached from three points of view.

1. What usefulness is the procedure likely to have in children's experience for numbers of a size which cannot be conveniently dealt with by mental arithmetic?

2. Can the children themselves make an experimental approach to the procedure?
3. How can the procedure be analysed and its elements also considered from the point of view of usefulness to the child?

One other general consideration remains to be mentioned. Children who get their reckoning right often fervently enjoy their success. Though these children have a most valuable experience of a skill and a mastery which comes with practice, only rarely do they also enjoy, in our present ways of working, the pleasure of mathematical discovery. Only too often some of their pleasure in 'sums' arises from the fact that in some types of school the arithmetic lesson is one of the few in which able children can get on with a job of sorts instead of listening to others talking. Also the artificial values given to success in reckoning by public opinion make it difficult to distinguish pleasure in success from a real liking for arithmetic. Even future mathematicians may not benefit from early specialization on mere computation. However, when computation is undertaken by any child, able or otherwise, no teaching will be satisfactory that does not include real mastery by the pupil and the joy of success.

How far many children, even among those of more than average ability, fall short of happy success is seldom appreciated. The rules of computation are full of details which, if they are taught at early ages before procedures seem a natural consequence of number behaviour, are difficult to teach except authoritatively. Rosie in Chapter VII shows the absurdity of a partly remembered ritual. Abler children might feel more frustrated by forgetfulness than Rosie but share with her the passive attitude to arithmetic that they were learned by doing as you were told over and over again.

Addition

It is not hard to find examples within children's experience of the need for addition of a pair of numbers too large to be conveniently added mentally, though actually addition with shillings and pence is a commoner occurrence than addition with tens and units. For convenience, compound rules will be considered separately and after the simple rules, but much calculation with money and measures may precede the formal rules considered in this chapter and be part of the preparation for them. There is no difference in principle between addition with shillings and pence and addition with tens and units. The difference lies in the particular groupings, twelves or tens, and in the notation used.

The sum of several numbers of one digit is a useful preliminary to addition of numbers of two digits. There are plenty of opportunities, including addition of small scores in games and small costs. Skills required for these additions are much the same as for addition with tens and units.

Examples of addition with tens and units might arise in returns for milk or attendance for a pair of classes or a pair of days. The numbers involved might be 35 and 42 or the more difficult case of 38 and 46.

A first essential for dealing with these is a readiness to break up any one of the numbers, e.g. 35, into 3 tens and 5 units. The connexion between taking a token to stand for a group and this writing of numbers in tens and units is dealt with in Chapter II. A practical illustration of a tens 'token' is the ten-shilling note, but it is recommended that apparatus of a standardized kind should be available for dealing with tens and units. A red counter for ten and a white one for units would be nearer to a complete symbolization than, what is probably preferable in use, some token like a bundle of sticks, a bar of beads, a card which does or could hold ten spots or squares, which shows the actual ten. The building up of 35 from 3 tens and 5 is of rather more common occurrence than the breaking down of the 35 which is the first requirement in written addition. Closely associated with this very important step of thinking of the two numbers to be added in two parts each, tens and units separately, is the rather more subtle idea that the order of the addition of the four resulting parts does not matter. There is a stage at which this may present a difficulty, i.e. a stage at which it is not clear that (3 tens+5 units)+(4 tens+2 units) is the same as (5 units+

2 units)+(3 tens+4 tens). It is important that this stage should have been passed before tens and units addition is undertaken. It need not be attacked directly, but note should be taken of the flexibility which children show in dealing with the sum of several small numbers. The other elements required are the items of the addition table, and, if several numbers are to be added, the extension of the addition table to bonds in which one of the pair is increased by a number of tens, i.e. not only 8+6 but also 18+6, 28+6, &c. It would be well if the addition bonds were known, through experience and some direct practice, well enough for counting in ones to be resorted to only as an occasional verification. It would, however, be possible for an experimental stage of tens and units addition to occur much earlier and for the final stage of practising a rule to await rather greater accomplishment in dealing with small numbers. In the examples under consideration, viz. attendances of 35 and 42 or attendances of 38 and 46, the experimental stage would involve using apparatus to represent the numbers of attendances and regrouping the tens and the units for the total. In the second case the 8 and 6 would produce enough to exchange 10 units for an extra ten bundle. The order of operations might not be the conventional one of adding units first; indeed if the sum be worked mentally the efficient order of working 38+46 is

38 and 4 tens = 78
78 and 6 units = 84

thus adding tens before units. It is when the experimental methods are gathered up into a form convenient for working without apparatus that the possibility of having extra tens from the total of the units makes it worth while to decide on the units as the first to be added. Particular cases can be dealt with experimentally until confidence in the usefulness of the process and in its elements is such that it can be profitably practised as a whole.

Addition does not as a rule create any great difficulties, but it shares some of its characteristics with other processes and it is therefore important that none of its elements should be slurred over. When addition of hundreds, tens, and units is required, or even an answer in hundreds, it is important to be sure that the use of the nought as a place-holder is well established. The understanding of place value and the use of a symbol, nought, for an empty place is a vital part of our methods of computation, which arise from our way of writing numbers. The idea is a sophisticated one and sufficient in itself to justify the plea that learning to read, write, and use larger numbers, is enough in itself for quite a long time before we begin to calculate with them.

Subtraction

As soon as larger numbers are used for measuring, say for measuring in inches within the range of the dressmaker's tape, problems of comparison may arise, and there may come a time when counting on in ones is an unduly clumsy process. It must be supposed, too, that the idea of a difference has emerged from this complementary addition and that the number elements of the situation can be written down in the form 54—38. All these suppositions should make written subtraction quite a late-comer to written computation; but even before these conditions are fulfilled particular problems may be solved experimentally by using apparatus for the numbers involved, e.g. by showing 54 as 5 tens bundles and 4 units and 38 as 3 tens bundles and 8 units. It is unlikely, however, that experiments with apparatus will follow the rule for computation, which is often favoured, in which extra tens bundles are introduced to overcome the difficulty of subtracting 8 units when there are only 4 units in the larger number, apart from the tens. What the child does may depend on the circumstances in which the need for a difference arises. If the problem is a comparison, he may show both numbers with his apparatus, 5 tens bundles and 4 units and 3 tens bundles and 8 units; if the problem is one of taking away or if he is already able to disregard the type of problem and use apparatus merely to deal with notational difficulties he will probably set out 5 tens bundles and

4 units and proceed to remove enough bundles and enough units to make up 3 tens bundles and 8 units. This will almost surely lead him to break up a tens bundle to make his 8 units, in fact it will lead to what is known in computational method as the method of decomposition. So long as the work remains on a purely experimental and occasional level and the use of apparatus is not standardized in any way, no habits will be formed, and future teaching will not be prejudiced. As experiment proceeds there will be practice in representing tens and units and, as in addition, in dealing with the tens and units separately in whatever order is convenient. At some time (better late than early), it may, however, be decided that problems involving such subtractions are of frequent occurrence and that the subtraction form of the addition-subtraction table is well enough known, in fact that the time has come for this rule of reckoning. If now it is decided that equal additions is the method aimed at, the fact has to be faced that this involves a purely artificial device. Stated in adult terms the device is 'If equals be added to each of two numbers (in this case a ten to each) their difference is unaltered'. Such a statement is of course inappropriate for a child, though the device can be sanctioned in a child's terms, and it can be justified by its success once the device has been taught. The question is really part of a much larger one, namely, What kind of arithmetic shall be taught to children under 12? But, so long as there are any children of primary school age acquiring mental habits associated with what is said to be the most accurate way of performing subtraction, it is not inappropriate to examine in detail what is involved in the method and what preparatory experience may help them to view this particular device as a sensible as well as a convenient trouble-saver. An example of equal additions which has real significance is available. Sheila, aged 4, realized that her age and her sister's age increased year by year, but her sister remained 3 years older, 7—4, 8—5, 9—6, &c. It is no long flight of fancy for an older child to appreciate that his age, now 9, will be 19 in ten years' time, and his brother, now 14, will be 24 in ten years, the difference between their

ages being 5 years in each case. This is a good example of the sanction given to an arbitrary adjustment by some link with reality. The difficulty of equal additions, that is, of saying that 54−38 can be treated best thus as the difference between

64 (which is 5 tens and 4 *units and* 1 *ten*)

and 48 (which is 3 *tens and* 1 *ten* and 8 units)

is, as has been stated, that it is artificial. If, however, it be likened to a difference of age

54 years and 38 years now

or 64 years and 48 years ten years hence

the only remaining point is the use of the ten with the units in one line and with the tens in the other. The illuminating example, that of difference in age, would of course be introduced before tackling the rule, so that reference to it would be immediate at the critical point of using an extra tens bundle instead of using one ten from the fifty as in the method of 'decomposition'. If the ten is introduced while apparatus is still present, the placing of it with the units in one case and with the tens in the other is less troublesome.

Doubtless other approaches to this difficult rule exist, but if it is to be tackled by primary school children as anything but an arbitrary trick, real sanctions are necessary. The rule does not seem appropriate at the early ages at which it is often taught. Even on the grounds of occurrence, large number differences are not very likely in the experience of younger juniors. Mental methods suffice when the difference is small even if the numbers themselves are large. If there is a difference of only one between the tens, e.g. in 45−38, the difference falls within subtraction form of the extended addition table 38+7 = 45.

Multiplication

This process suffers very greatly from lack of readiness amongst those who learn it. It is often taught before even the multiplication table has been fully accepted as a usable accomplishment.

> Beryl, for instance, was already 13 years old, a pleasant, rather slow-minded girl, who had spent her primary school arithmetic lessons unsuccessfully trying to learn rules of reckoning. In the secondary school she had learned to make correct responses to multiplication tables, and she could set out multiplication of a number of two or three digits by a number of one or two digits correctly. She also knew how many half-crowns there were in £1 and other useful facts about money. She was asked to write down how many half-crowns there were in £20. Her answer was 152. She had added eights, but broken down at the nineteenth.

It is not certain which of several difficulties had defeated her. She may not have recognized the situation as one in which multiplication might with advantage be used; if so, she failed in a first principle of readiness for learning tables. She may have thought the multiplication too hard to perform mentally, but 20 eights is a notational extension of 2 eights which is of fundamental importance to 'long multiplication'; for instance, if we have 48 × 23, the very first partial product is 8 × 20. Again $20 \times 8 = 8 \times 20$ is number knowledge which it is well to acquire before embarking on formal rules of multiplication, the sanctions in experience for this flexibility with factors have already been discussed. If Beryl had added twenties instead of eights she would have been less likely to lose count. This single wrong answer disclosed an insecurity about fundamental elements which would have remained undetected had her group counting been correct.

The extension of the multiplication table to cover such products as 8 × 20, 80 × 2, and 80 × 20 when 8 × 2 is known is an item essential to co-operative understanding of rules of multiplication. For certain products it can quite easily be given the sanction of usefulness. To find the number of shillings in a number of pounds or of minutes in a number of hours involves such products. When a few such products have been found and noted, the pattern of the numbers emerges

£4 80 shillings c.f. $4\times 2 = 8$.
£6 120 ,, c.f. $6\times 2 = 12$.
£9 180 ,, c.f. $9\times 2 = 18$.

Or if flexibility with factors has reached the stage of breaking up numbers into factors and altering the order of continued products, then

$$4\times 20 = 4\times 2\times 10 = 8\times 10 = 80$$

or

$$\begin{aligned} 40\times 20 &= 4\times 10\times 2\times 10 \\ &= 4\times 2\times 10\times 10 = 8\times 100 = 800. \end{aligned}$$

All this and more is an essential part of finding such a product as 48×23, which is not difficult if the items are allowed time and opportunity to become second nature through usage before the rule is under consideration.

There is, however, an additional item, which is generally taken for granted, but which needs considerable experience before it is acceptable. It is one of those things which should not be dealt with explicitly but which should be given practical sanction incidentally and without comment. The product 48×3 has to be treated in two parts 8×3 and 40×3; these two parts are then added. That is

$$48\times 3 = (8+40)\times 3 = (8\times 3)+(40\times 3).$$

The final addition is done mentally. That the product required is the sum of the partial products is not self-evident to children.

> Brenda and Polly, aged nearly 9, had learned how to multiply 48 by 3, though they did not know their tables very well. They were doing 'practical arithmetic' in the quadrangle, using a yard-stick to measure the length of brick edging to the flower beds by marking it lightly in chalk yard by yard. They had marked 43 yards. The yard-stick was coloured red, white, and blue, in feet. When Brenda was asked how many feet of edging there was, she began to count in threes. She was asked if she

> knew how many feet she would have up to the 10-yard mark. She did not know, but Polly knew 10 threes was 30. Polly then achieved 20 threes as 60 by addition and from that she managed 40 threes as 120. To take the step from 40 threes to 43 threes was beyond even Polly until she actually stood by the marks of these last three yards. Brenda afterwards managed to work other numbers of yards without counting threes when she found their place in her marked measured edging.

The marking of the yards in a line of edging gave a physical sanction to the additions both of the $40+3$ and the $120+9$.

When the multiplier is of two digits as in 48×23 the need for dealing with a sum of partial products, that is, the 'distributive' nature of the problem, is more obvious, for 48 is multiplied by 20 and then by 3. The results, in separate lines, are then added.

To summarize

1. Multiplication as a process first has to be understood for small numbers and then extended to larger ones. The difficulty of extending 'multiplication problems' to larger numbers is often not what it seems. It may be that with small numbers problems are, in fact, solved without the use of multiplication (as in Fig. 16).

2. The multiplication tables and their notational extensions are elements of the process as a whole requiring sanctions in experience.

3. It is also necessary to be able to split a number into its notational parts, as in addition,

e.g. $48 = 4$ tens and 8
$23 = 2$ tens and 3.

4. 'Commutative' and 'distributive' laws must be taken for granted from experience of how numbers behave in particular situations

e.g. $48\times 23 = 23\times 48$ (commutative)
and $48\times 3 = 8\times 3+40\times 3$ (distributive)
or $48\times 23 = 48\times 20+48\times 3$.

5. Particular cases of multiplication may be approached with apparatus experimentally if they arise at an age when mere group counting is becoming undesirable but readiness for rules is not complete. An example has already been given of quite young children wanting to know 14 × 7 inches when using a measured shoe as a measuring rod. These children were too young for anything but a layout of 14 groups of seven. At a later stage, however, it might have been noted that 7 groups of fourteen was an alternative to 14 groups of seven and the fourteens could have been set down quickly by using tens apparatus, say tens bundles, as in Fig. 35.

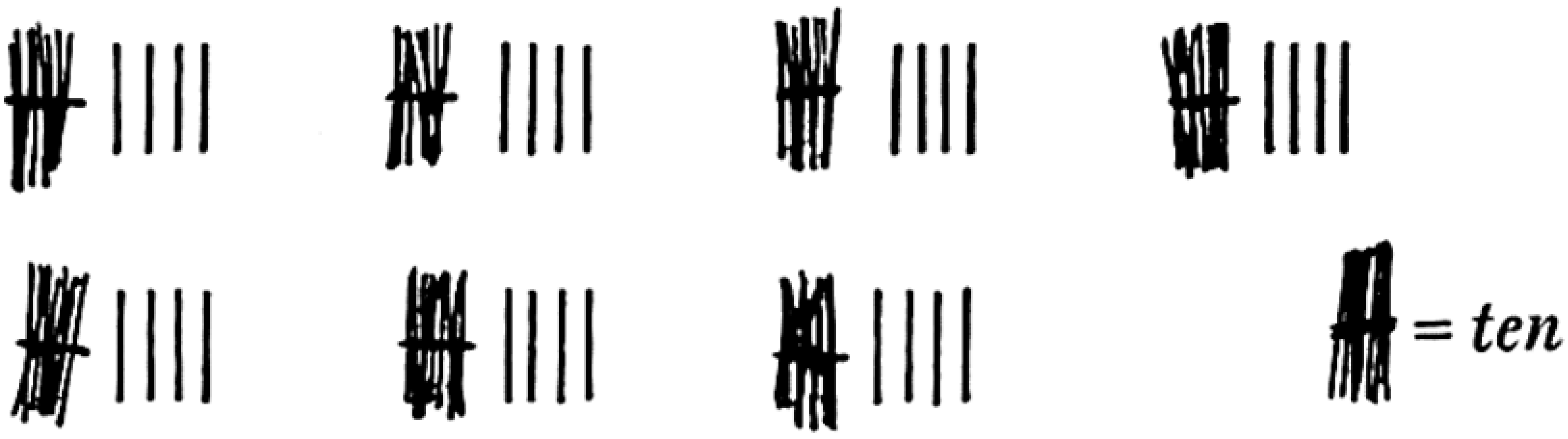


FIG. 35. Tens and units apparatus for 14 × 7.

It would be apparent that 7 tens and 7 fours were involved. Such an experimental approach is in fact a demonstration of the distributive part of the rule for multiplication, and therefore takes its place as one item in preparation for learning the whole process with co-operative understanding. It is wise and common practice to deal with simple cases first and with increasing difficulties stage by stage. Fully experimental working may well only be applied to early stages and to the elements of the process. Explanations of what has been taught arbitrarily are, however, to be avoided; elaborate apparatus is sometimes designed for this purpose.

Division

Situations involving division of larger numbers outside the range of the multiplication-division table are not very likely to occur embarrassingly early. There is a lot of preliminary work to be done within the table's range including the preparatory work on the meaning of remainders in different kinds of situation. Something has been said in Chapter V about the different types of situation in which division arises, and the point is returned to again in Chapter X. Briefly, the matters to be learned from experience with small numbers before there is much chance of division with larger numbers having significance are:

1. 'Sharing', that is, finding the size of a share or group when the number of equal shares is known;
2. 'Measuring' or 'Comparison by ratio', that is, finding how many times one magnitude or group is contained in another;

and in a different category,

3. Appreciation of the related nature of the multiplication-division processes.

The earliest examples outside the range of tables are likely to be those in which the divisor is small, e.g. 3, and the dividend rather more than 12×3. If the work on the notational extensions of the multiplication-division table has been well done it will quickly be realized that numbers between 30 and 60 contain between 10 and 20 threes and so on. Experiment may be needed before the exact number is determined in, say, 42 by dealing first with the threes in 30 and then with the threes in 42−30. It might well be that 36 was taken as a known number of threes and the count taken on from 36 to 42. A larger number like 75 yields more readily to the general method of dealing first with 60 and then with 75−60. That the number can be dealt with in two parts like this needs a careful approach, as in multiplication. Particular cases treated experimentally lead to general acceptance, especially if multiplication is well established. An example can be offered of the power of motive in producing readiness for learning a difficult process.

Some children who visited a cattle farm were horrified by the high cost of feeding stuffs which the farmer mentioned. It was suggested that he had a large herd to feed, and the children asked how much each cow cost in a year. They were told to

> find out from the amount given for the herd, and they counted the cows. This presented them with a problem which they set to work to solve by trial, building up their result by all sorts of devices, including multiplication by 2 and 10 of results proved to be much too small. Finally the trial divisors of long division were shown to them as a more methodical approach, and the whole process of long division was established within a remarkably short time.

It would not generally be wise to start, as these children did, with a difficult case, except under the influence of a strong motive; but some measure of trial and error in particular situations which are beyond the normal scope of mental arithmetic is a healthy introduction to orderly approach.

Changing from a number of small units to a smaller number of large units is a type of situation very likely to involve division, just as the converse involves multiplication, so that division by small divisors is quite likely to occur in dealing with measures. 'Short division' is often taught by what is called the Long Division Method, i.e. by writing out each partial divisor in full. One reason for this is directly related to preparedness in table work, in particular knowledge of the extended addition-subtraction table. It is worth analysing a short division example to show the number of mental skills which may be involved, e.g. in $334 \div 7$ (a difficult example has been chosen to show repeated difficulties). It is first necessary to use the extended division table to determine the number of sevens in 330, though this is quickly reduced to

$33 \div 7$ It is first noted that 28 comes in the table of sevens and then that 28 is 4 sevens.

$33 - 28$ must be computed. This is an extension of $13 - 8 = 5$. The 5 is then interpreted as 50 and combined with 4 units.

$54 \div 7$ must next be noted. 49 is in the table of sevens, and is 7 sevens.

Lastly $54 - 49 = 5$ which is an extension of $14 - 9 = 5$.

If the division be written out in full in either of these forms:

$$\begin{array}{r} 47 \\ 7\overline{)334} \\ \underline{280} \\ 54 \\ \underline{49} \\ 5 \end{array} \qquad\qquad \begin{array}{r} 47 \\ 7\overline{)334} \\ \underline{28} \\ 54 \\ \underline{49} \\ 5 \end{array}$$

the difficult subtractions are written down, whereas in

$$\begin{array}{l} 47 \text{ and 5 over} \\ 7\overline{)334} \end{array}$$

the 28 of 33−28 is not written and neither 54 nor 49 of 54−49 appears on paper. The recognition of the 4 in the quotient as 4 tens is helped by the first form of the long division, even if the writing in of the nought in 280 is afterwards abandoned.

The great importance of preliminary work on how we write numbers in tens and units and the need for flexibility in splitting a number into its component parts in this tens grouping is more than ever apparent in dealing with division. It is not surprising that so many children who learn these rules early in the primary stage only accept them as arbitrarily taught procedures. It is not suggested, however, that, even at later ages, any general reasoning should be attempted. Careful consideration of the elements of skill involved, practical sanctions for these elements, and, above all, usableness of the whole procedure are the important characteristics of readiness. Re-examination when all is secure is another matter, but this lies well outside the primary school range, even for unusually able children, except in so far as they do it for themselves as a natural outcome of mastery.

Much that has been said constitutes an attack on existing practice, for which causes external to the primary school are in part responsible. The years between 7 and 9 are probably the most critical for the majority of children. It is in these years that children of average and good intelligence should be obtaining a lively appreciation of how small numbers behave in circumstances within their experience. Experience of measures, with computation still at the mental level, is most fruitful as a preliminary to written arithmetic, as well as being appropriate to young children's activities.

The compound rules are dealt with as part of the next chapter on extended experience of money, weights, and measures with appropriate computation.

It is often said that alternative methods and short methods of performing fundamental calculations should be avoided or reserved for abler children. Many of the short methods depend on a different application of the distributive law which is characteristic of number behaviour and which is implicit in some of the general rules. It is not surprising, therefore, that children in a primary school, who have learned, for example, to multiply without any feeling for the appropriateness of adding the partial products, should fail to adapt this to special combinations of other products. For instance, when multiplying 248 by 23 the partial products of 248 by 20 and 248 by 3 are added; but in multiplying by 198, instead of adding three partial products, it is shorter to multiply by 200 and by 2 and subtract. It is possible, also, that multiplication by 200 is considered difficult when it stands alone instead of as part of the standardized pattern of a rule, though acceptance of such a multiplier as being no harder than multiplying by 2 is really a prerequisite of learning the rule with co-operative understanding.

Again flexibility with factors of a continued product and extension of this to mixed multiplication and division are the basis of other short methods, for example, multiplying by 25 by using the equivalent multiplication by 100 and division by 4. Confidence in this sort of number behaviour is best gained with small numbers for whose various groupings some real sanction can be found. For example, equivalents in pence, sixpences, threepences, and halfpence can be found for

$$12 = 2 \times 6 = 2 \times 2 \times 3 = 4 \times 3 = 3 \times 4 = \frac{24}{2}.$$

Finally it must be emphasized again that what has been said about elements of preparedness for learning a rule does not mean that these elements should be taught systematically to young children for the sake of making them ready to deal with a rule at a particular time. Direct approach would frequently defeat its own ends, and it is for the teacher to take opportunity as it occurs over several years of learning from experience. If the teacher is aware of what is of special importance, a rich experience and some postponement of formal rules will give children time to acquire the necessary background. In particular, items of number behaviour, which have in this chapter been given their formal names of commutative and distribution laws, will be taken for granted by children from their experience, both practical and numerical, without ever being formulated. The teacher of insight will, however, note when children have reached these stages of mastery. An experimental approach to reckoning, as need for it arises, will often provide some of the necessary experience of this number behaviour.

IX

MONEY, WEIGHTS, AND MEASURES

ONE of the most fruitful contemporary trends in the teaching of arithmetic is the increasing prominence given to extension of experience connected with money, weights, and measures. It is recognized that this is in keeping with children's interest in everyday things and that it gives them opportunity for active doing as well as figuring. It is not so generally appreciated that the mental computation involved in work with British weights and measures gives children a magnificent experience of small number behaviour in a valuable and interesting context. When number is applied to numbering units of length, there is the added mathematical advantage of making a beginning in the association of numbers with a linear scale.

Generally, practical work with money, weights, and measures and common-sense applications of them are allowed to run concurrently with written reckoning with tens and units during early years in a junior school. Much of this practical work could well take precedence over any formal reckoning and might indeed provide motive for it as numbers become larger and mental arithmetic more sure. The experimental approach to written reckoning advocated in the previous chapter would be in keeping with this general practical attack. For example, the need to change from one unit to another smaller or larger unit is an excellent opportunity for experiencing a need for multiplication or division, either on the level of mental or written reckoning.

Money

Shopping is one of the most familiar of everyday happenings, except in the depths of the country. It is encouraged in schools as an imitative and imaginative activity at an early age, and it continues in those schools which encourage practical work connected with arithmetic far into school life, at least for less able children. Abler children may quickly outgrow the familiar practical setting of the classroom shop and be better employed dealing with its reckoning in a setting of their own imagination, unless they are confronted with some new condition in this most variable of activities which makes them glad to obtain a new practical sanction for their thinking. Shopping is so varied that the conditions which may arise are worth summarizing from the point of view of classroom practice.

1. Variation on the informative side.
    - Current prices, rationing, &c.
    - Different types of shop, the grocer, the draper, the furniture dealer, &c.
    - The café, the market, the fair.
    - The Post Office.
    - The Station.
    - The travelling salesman.
2. Variation in the reckoning involved.
    - (*a*) The Buyer.
        - Range of prices.
        - Smaller and larger sums to spend.
        - Number of items bought at a time, checking bills.
        - Checking change.
        - Costing with quantities, e.g. 3 yards at so much a yard, &c.
    - (*b*) The Shopkeeper.
        - As for the buyer but with harder possibilities.
        - Making bills and giving change.
        - Measuring amounts by weight or length.
        - Using ready reckoners.
        - Checking the stock and the till.
3. Variation in class management.
    - Individual, e.g. a shopkeeper making up orders.
    - A small group (often the best arrangement).
    - Groups with equal numbers of shop assistants and/or shops.

Family groups with errands.
The whole class, very seldom except for discussion of possibilities.

4. Variation in use of apparatus.
   Coins.
   Weights and measures.
   Use of cartons made up to correct weights.
   Different degrees of realism in the activity.

Within this framework there is great scope for dealing with children's developing interests and with their differences of ability, as well as with their growing powers of dealing with numbers.

The early stages of dealing with coins as tokens for groups of coins of smaller value has already been discussed. The exchange of coins without change of value must be extended until considerable facility is gained up to the value of £1. The parts of a pound, using perhaps simple fractional naming, may precede formal consideration of fractions.

Considerable practice in exchange between shillings and pence while the practical aspects of the work are in evidence is a guarantee against all but careless mistakes of using the familiar 10 instead of an equally familiar 12 when the stage of more paper work arrives. The noting on paper of sums spent in the form of a bill arises almost spontaneously as a matter of experience, and the first dealing with bills may be quite experimental; the noting on paper is a convenience as a record rather than as an aid to reckoning. Bills may easily occur in experience before the children are sufficiently accomplished in mental arithmetic to make it suitable for them to practice adding small sums of money and so acquiring skill in addition. Postponement of formal practice of written bills may be important for other reasons. There are certain skills in dealing mentally with small sums of money which may not be those used when a great number of items are written on paper. In dealing mentally with two sums of money only, the shillings can be added before the pence, and if there is a complete extra shilling in the addition of the pennies, various short cuts based on intermediate coins may take the place of a more general method used in a longer total of pence. The learning of a so-called pence table, proceeding in steps of 10 pence, does not seem worth while if proper practice is given as required in the table of twelves and in the extended subtraction table, that is, in finding the difference between, for example, 57 and 48 when the difference between 17 and 8 is known.

Giving change in approved fashion is, of course, a process of complementary addition appropriate to mental and practical work; it can be treated independently of any skill which is practised as the best means, after experimental trials, of dealing with a difference between two sums of money large enough to deserve recording on paper. Such a need is unlikely to occur in the early years of dealing with money, unless it is introduced by an arbitrary syllabus.

Changes between shillings and pounds are sometimes taught by using the ten-shilling note as an intervening step. The skill involved in dealing with 20 as multiplier or divisor is so important that it is worth special attention, both as a mere skill and also as a skill founded on clear appreciation of the number behaviour involved. The association with money gives the skill its sanction and purpose. Multiplication by 20 is worth time spent on experiment and clarification of experiment, followed by practice.

A few special skills in mental reckoning with money are appropriate for many children in the primary school, though the less able children may well leave special devices to a later stage. Shopping may demand the costs of dozens with its easy change from numbers of pence each to numbers of shillings per dozen, and the cost of an ounce can be easily reckoned if the cost of a pound, that is 16 ounces, is a multiple of 16 pence. The corresponding short methods depending on 20*s*. in £1, 20 cwt. in 1 ton, and 20 in a score may also be used at later stages of primary work.

Although mental reckoning applied to money and experimental stages of written work may precede the written reckoning with tens and units described in Chapter VIII, most of the written work involving pounds, shillings, and pence is, with the possible exception of addition, best left until the corresponding rule for tens and units is established. The money rules are the same in principle as the rules for tens and units, but the break from groups of 10 to groups of 12 or 20 upsets the rhythm of work and makes them harder to set down tidily. Multiplication and division for awkward sums of money or larger operators may well be omitted from the primary school course. It has already been suggested that able children will find ways out for themselves and it might be added that any child accustomed to treating practical and numerical situations by common sense will find means, even if those means are primitive, to deal with situations in his experience, for example, the cost of rail fares for a party, in which the neatest solution is outside the range of his formal skills. Repeated addition and subtraction, for example, are only to be despised when the range of skills includes something better.

Surprising multiplications sometimes occur spontaneously on lines equivalent to the building up of ready reckoners. This tabular form of dealing with calculations which are wanted repeatedly is most instructive. It often includes interesting additive devices based on listing appropriate single figure multipliers and successive tens.

Examples of costing quantities involving units of length, weight, &c., are given in the appropriate paragraphs for measures.

The emphasis on background of experience contributing to mental rather than written arithmetic in dealing with money at the early stages, if it needs justification, might find it in the extraordinary incompetence of many adults in dealing with this reckoning. Inability is occasionally observed even among those for whom reckoning is a part of their work.

A ticket conductor on a river steamer with a fare of 8*d*. for the trip was unable to calculate the cost for a party of 23 without counting the tickets 3 by 3 as the charge was paid 2*s*. by 2*s*.

A shop assistant, who was counting a part of a bill in ones, explained that she always did this if the number of pence exceeded 1*s*.

Another shop assistant, who was finding the cost of 4 cups and saucers at 2*s*. 9*d*. each, first added 2*s*. 9*d*. to 2*s*. 9*d*. and then 5*s*. 6*d*. to 5*s*. 6*d*. She explained, when asked, that she had learned multiplication at school but she felt much safer with addition.

This lack of confidence at a later stage is the kind of reaction to be expected when a process has been learned by rote without adequate sanctions in use.

On the other hand most workers using particular forms of calculation continually, especially if these are at all complicated, learn on the job the most appropriate technique, and so acquire skill. Indeed, adaptations to particular calculations in employment are often reported of those of little ability. This is sometimes true of children taking part in family concerns out of school.

Weights and Measures

It is very seldom even in adult experience that more than two units of the same kind of measure are in use together. For instance, hours and minutes or minutes and seconds, may be used in stating a time, but seldom all three together. The long formal examples on reduction and the four rules with quantities stated in terms of several units are less common in primary schools than they were, and common sense might have banished them altogether were it not for the influences of tradition and external examinations. The kind of work with weights and measures which is taking their place consists of practical experience of the measures themselves and of small enterprises which simulate the everyday uses of these measures. These include data collected by the children or taken from original sources, such as railway time-tables, post office regulations, and so on. As with money, the rules do not differ in principle from the four rules for hundreds, tens, and units. A matter of first importance is a clear understanding of the relation between two units which are likely to be used together. A knowledge of tables of weights and measures is built up on an experience of the use of the units. This is likely to assist both in memorizing results and also in estimating measures and choosing sensible units for practical purposes. Many schools are equipping themselves with a sufficient variety of measuring tools, the simpler ones home-made, to give children skill in measuring as well as skill in simple computation involving measures. Introductory work in weights and measures is discussed in Chapter III.

Weight

When ideas of weight and a unit of weight have been acquired by early experiences with a beam balance, children generally readily accept other forms of balance familiar in experience; but inquiries about some of the stranger varieties, such as the weighing of an obviously heavy object by moving a small weight along an arm, are common in the later junior stage. Whenever they occur, some kind of equivalent experience should be devised. The see-saw will dispose qualitatively of some of the problems of leverage, and closer observation of the construction of the school weighing machine with its system of levers or the double arm of the grocer's balance may satisfy questions of how, if not of why. Spring balances can be graduated by using known weights. Their graduated scales are helpful in dealing with simple fractional parts of a pound, as a movement registering one pound is clearly equivalent to four times the movement registering the quarter, whether on a dial or linear scale.

It is important at early stages to have enough weights for use on a beam balance or a grocer's balance. There should be 16 ounce weights, 8 two-ounce weights, 4 four-ounce weights, and 2 half-pounds. Afterwards, the grocer's pyramid consisting of one of each of these together with 1 pound can be discovered by experiment to be sufficient to weigh anything up to 1 lb. 15 oz. There is a similar set of weights that will be sufficient for any number of pounds up to a stone. A brick can be made the basis of a seven-pound weight, and there are several objects commercially available and conveniently packed which weigh approximately one ounce. The larger weights in common and commercial use, the stone, hundredweight, and ton, can be related to observation if not to experience. The stone as a common unit in which to measure human weight is not difficult, and the recording of the weight of members of the class is a usual basis for work with this measure. The weighing and grading of garden produce such as potatoes might form a good experience of linking the pound and the stone.

The hundredweight is commonly dealt with by reference to a sack of coals with 20 sacks to the ton. Costing exercises are often taken as a means of lingering over the consideration of a measure barely within the children's experience. The ton, however, is more likely to appear spontaneously in the children's field of inquiry, if not of their experience. Any train or lorry enthusiast may notice the loading limit written on the side of a truck, and similarly some story of the sea may mention the tonnage of ships, a more difficult notion which may lead to scientific as well as purely mathematical inquiry. A visit to a weighing station for lorries is as good a practical experience of these larger weights as a junior pupil is likely to acquire.

Most of the numerical work with direct relation to the children's experience deals with numbers of pounds and ounces and with possible costing at so much per pound. Familiarity with counting in fours, eights, and sixteens up to 80 is quite a useful accomplishment. If abler children use their facility with number to relate larger measures to much smaller ones, the numbers 112 and 2,240, connecting pounds with hundredweights and tons, are numbers which should be known, not only as they stand but as continued products of their more important factors.

Liquid Measure

Formal processes are quite out of place is dealing with these homely measures. Real familiarity with vessels holding a pint, a quart, and a gallon is the basis for a firmly fixed table of these measures. Unfortunately, since this is a volume measure, the eye is not a trustworthy guide to quantity, and pictorial diagrams illustrating pints, quarts, and gallons are often misleading; they either show distorted-looking vessels or do not give a correct representation of the volumes. The only convincing method to show that a small bucket holds 4 quarts or a gallon is to use a quart measure to fill it.

It is convenient general knowledge attractive to children to know something of the approximate capacities of common vessels such as cups and kettles, and perhaps also to consider smaller measures like spoonsful in relation to them.

The link between liquid measure and other measures of volume or the weights of measures of particular liquids is as a rule beyond the range of young children.

Work involving costing of measures, of, for example, milk, is within everyday experience. A real or imaginary record of the milkman's sales in a row of houses with quantities and costs might be the limit of useful computation.

Length

The measuring of length plays a great part in the appreciation of number and in later mathematics, as well as being of practical importance in itself. Lengths are used to represent other quantities and almost as symbols for numbers, especially in the later stages when fractions, signed numbers, and so on are taken into consideration. At this stage only the early practical and useful developments in measuring lengths are in question, but it is of mathematical importance that these should be treated carefully and accuracy encouraged.

First experiences in the choice of units and the use of yard, foot, and inch are given in Chapter III. The visual appeal made by the lengths themselves and easy measuring apparatus such as variously graduated yard-sticks helps to fix the numerical relations between these small units. A demand for some smaller unit than the inch arises even at the infant stage if shortish lengths are being measured. Sub-measures named by numbering the parts into which the inch is divided, for example 4, denoted by $\frac{1}{4}$-inches for the unit and $\frac{3}{4}$-inches for 3 of these units, give no trouble, though it is a pity that the verbal naming of a 2-part and a 4-part unit does not conform with naming 'thirds' for a 3-part unit. The words half and quarter are sometimes used loosely for any 2 parts and 4 parts instead of for equal unit parts. The demand for smaller units and therefore closer accuracy in measuring is a fundamental development, which demands more attention than it receives. It is a step towards mathematical ideas of what measurement involves, and it is bound up with children's practical common sense that when they are using a yard-stick for measuring the distance along a corridor a little bit of an inch does not matter very much. Preoccupation with exactitude in measuring and distress that it cannot be just right is not altogether natural. It arises from the emphasis placed on computation which must be correct; the argument may go: 'Measuring is something we do in arithmetic; in arithmetic what we do is either right or wrong; measuring must be just right.' It is as important to recognize that measuring is one of those activities in which we get things as good as we can with the means available, or good enough for our purposes having chosen suitable tools, as it is to recognize that reckoning has a right answer. This kind of discrimination develops if work of both kinds is purposeful and within the scope of the children's experience. It may be helped by such practices as estimation. To estimate a length uses the eye as a rough measuring instrument; to follow this up with something more reliable and to compare the results, noting the excess or defect of the estimate, is just a first stage in refinement of measuring.

Rulers showing various sub-divisions of an inch on different edges are too complicated for early use.

Some simple outdoor measuring involving a large number of yards is quite appropriate to almost any stage of junior school work. The pace as an informal unit can be used with or without much consideration of its variability, the length of an average pace and so on. Refinements can be adjusted to the age and ability of the children. Work of this kind may link up well with the reading and making of plans and maps. Yard-sticks are generally unsuitable for measuring longer distances, and cords stretched from point to point can be used even before the larger unit, a chain, is introduced. Practice in estimating longer distances may well suggest the convenience of having a longer unit than a yard by the realization that for distances longer than a few yards the tendency is first to estimate a group of, for instance, 5 yards, and then use this like a known unit.

A simple measuring instrument very useful for longer distances, especially those which are not straight from point to point, is the trundle, a wheel with a handle, like a wheel-barrow without the barrow. The idea of a distance measured through contact with each point of the wheel in turn is not a difficult adaptation of common experience, and its applications to mechanical recordings of distance travelled also appeals. First experiments should probably be with wheels of any circumference before the use of a specially constructed trundle with circumference an exact number of standard units. The smallness of the diameter of a wheel of 1 foot circumference is generally a surprise, and there is a good deal of instructive fun to be had from estimating circumferences of wheels, hoops, and tin lids. A few able children may suspect the relationship of the diameter and circumference, or a relationship from wheel to wheel depending on the diameter. Such observation should not be forced at this stage but left to the stimulation of practical success. The difficulties of using this instrument are numerical rather than practical. Too many turns with a circumference roughly measured should be avoided.

Among the units of distance larger than 1 yard the one most commonly occurring in ordinary speech is the mile. But the mile is so large that it eludes visual estimation; its end is generally out of sight of its beginning, and even if a panorama is in view, foreshortening makes estimation a matter of difficult adjustment from informed experience.

First experience of the mile or a simple fraction of a mile is generally only a derived experience from the time taken to walk from point to point, assuming that a time of transit can generally be recorded, and that a normal speed of walking is felt by the child almost as a physical constant. In spite of its derived nature, this does give a point to point standard for the mile and a rough means of estimating comparable distances. It needs, however, the support at some stage of a direct measure of at least a known fraction of a mile. This can then be combined with a direct measure of time taken in walking to give an experience of the derived unit of speed.

A group of children of 10 years of age in a poor part of an industrial city made such a measure of half a mile along a street near the school, using two home-made chains of tape carried by teams of children. The children proceeded chain by chain, a team at a time, with the tapes carried shoulder high and extended fully as the leader marked the end of stretched-out tape. They noted points of local interest in passing, the butcher's shop, a side turning, a factory, the number of the house when the first furlong was reached, and so on. An illustrated account of their progress showed the interest of the butcher's queue and the passengers on the passing bus. The written account which accompanied the pictured panorama concluded with the name of the street terminating the half-mile, and the words, 'We then walked briskly back to school, arriving in time for mental arithmetic. We took 10 minutes to walk $\frac{1}{2}$ a mile, so our speed of walking was 1 mile in 20 minutes or 3 miles in 1 hour'.

The children had, of course, had previous experience of using their home-made chains.

The link, the chain, and the eighth of a mile form a very convenient group of measures, 100 links to 1 chain and 10 chains to 1 furlong or $\frac{1}{8}$ mile, in which the chain is the dominant measure to be related to experience as a known standard of length and basis of estimation. The relations between these measures and the yard and the mile are, as it were, well-worn side paths departing from the scale of 10 and introducing groups of 22 and 8. The inch, foot, and yard form one group, and the link, chain, and eighth of a mile or furlong another. The relation of the yard to the chain forms their connexion. As abler children push on with work involving measures, it may become convenient to use the number of yards in a mile, 1,760, as the remembered product of factors, 22, 10, and 8.

Computation arising in experience seldom involves more than one pair of units, for example, yards and feet, or feet and inches, or yards and inches. There is no need to learn as though it were a new rule how to negotiate one simple unit change. The important thing is that the change of units should, when required, be undertaken readily with co-operative understanding. Changing from one unit of length to another gives excellent number practice, with the advantage that the operation carries in its context its own visual aid to the meaning of what is being done. For example, at an early stage the measuring of the height of a table might be read off on a yard-stick marked in inches as 31 inches. If this measure is to be ranked among measures given in feet and inches the 31 must be placed in relation to the count 12, 24, 36 of inches in whole numbers of feet. The difference 31 − 24 is then found. At some stages this will be known, and at others perhaps counted in ones from 24; but whatever the mental processes, the yard-stick itself shows the relation between the 24, the 31, and the 36.

Time

Schools accept without too much analysis the view that ideas of time involve special difficulties. Teachers can take as their starting-point everyday usages and experience such as telling the time, using the calendar, noting the recurrence of day and night and of the seasons. This makes it unnecessary for them to pry deeply into the growth of children's ideas about the passage of time provided that they accept the need for patience in accustoming children to recording it and provided they do not substitute reckoning according to rules for facility in dealing with simple examples of using time records and making calculations. There are enough stages of difficulty in dealing with such realistic circumstances as intervals between times or dates to extend the powers of juniors over a wide range of age and ability.

> Paul, the youngest of three children whose mother was an astronomer, found the security which most children find in the routines of their own lives in the routines of the heavens. Precociously interested also in number, at 6 years old he made use of records of sunrise and sunset and of the phases of the moon. One day in distress he came to his mother complaining that the full moon was a crescent. She had forgotten to warn him of a lunar eclipse. Thereafter he joyfully added eclipses to his expectations and observations.

Telling the time by the clock involves experiences both of number and of position. A beginning is frequently made by infants, who may master some of the early stages of a long list of possibilities, many of which are given below.

The Clock

Position of numbers round the face.
Functions of small and large hands.
Hours and half-hours.
Quarter past and quarter to with attention to appearance of positions of the hands.
Counting minutes in fives.
Minutes past and minutes to; in fives; any number to 29.

Time-table time, i.e. minutes past up to 59.

Special attention to 15, 30, 45; 20, 40 (60) minutes past, with positions of the hands.

Practice in dealing with number of minutes not exact multiples of five.

Placing hour hand between the hours (a late and significant stage which involves co-ordinating two movements which have different speeds.)

The 24-hour method of reckoning time and its usages.

Intervals between times, without rules of thumb and with special attention to passing noon and midnight.

Applications to railway time-tables, radio programmes, &c.

Timing events, use of seconds hand, stop-watch, &c.

Other methods of timing and a simple historical approach to the measuring of time are of interest to older juniors. Making records of the movements of shadows can be linked with this in a very simple way.

> One class of 10 years of age had a small home-made water clock on which they could graduate the water-level by comparison with the classroom clock. They also had a pendulum bob which they adjusted to beat seconds, a sand 'egg-timer', and a candle into which they stuck pins, so that when they lighted it they could have an arithmetic race to see who could do most practice exercises before the pin dropped. They also timed their speed of writing to compare it with a record of words per minute by a skilled shorthand writer.

Ability to count seconds with fair precision also appeals, almost as a game, when there is any reason to be interested in timing a short period—for instance, in the timing of the interval between seeing the flash of lightning and hearing the thunder as a measure of the distance of the storm.

Reckoning of intervals between two dates is a useful type of calculation which requires practice because of the detail involved.

The numbering of the days and of the years is awkward because the same number applies to the whole of a day or year. It is, therefore, important to decide whether or no both end periods are to be included in the reckoning. The difference between two numbers marking two dates in the same month includes one end period only. The business of passing from month to month in the reckoning needs methodical work and knowledge of the calendar. The whole provides an excellent exercise in systematic, careful working. It may be noted in passing that the difference between numbers used as labels for, for example, a series of tickets or bank notes does not equal the number of tickets in the series; one more must be added for the ticket at the beginning. This is the same number problem as that for finding the number of days inclusive of given dates in the same month.

There are many other opportunities in everyday experience for extending the arithmetic of time and timing. Long examples of formal calculation of hours, minutes, and seconds are quite out of place, but it is convenient to be able to deal confidently with changing a small number of hours to minutes or a small number of minutes to seconds. This is an exercise in extending the multiplication table from say 6×5 to 60×5. The number 60 also is as a number worthy of study. Children are interested in its convenience because of its great number of factors as shown below and in its importance in history.

$$\begin{aligned} 60 &= 1\times 60\\ &= 2\times 30\\ &= 3\times 20\\ &= 4\times 15\\ &= 5\times 12\\ &= 6\times 10 \end{aligned}$$

But 60 does not have factors 7, 8, or 9. Children who have learned a little about angles are also interested in the fact that degrees, minutes, and seconds are used in angle measuring.

Area and Volume

The question whether measuring area and the use of cubic measure should ever be included in primary work depends more on the growth of geometrical ideas and on appropriate experience than on any ability to do the reckoning. A young child, who has (like Mary of Chapter V, page 68) made a meaningful approach to multiplication, can find how many square tiles with side one inch can be packed in the bottom of a box 8 inches long and 5 inches wide. The extension of this example to more awkward numbers involves no matter or principle. The idea of area and the choice of units of area and volume are more conveniently discussed along with geometrical ideas at the end of Chapter XI.

Speed

Children have experience of and interest in speed from a very early age.

> Rupert, aged 2, living within view of a busy railway line, would turn over in his sleep as trains passed, murmuring 'Express' or 'Goods' from the sound they made.

An experience of measuring speed is quoted in connexion with the measured half-mile in a preceding paragraph. Some experiments on distance travelled in given times, and times to go given distances, give precision to a common-sense approach by children to what is meant by travelling fast or slowly. Common speech settles miles per hour as the most normal derived unit for speed, but experiment almost inevitably involves smaller units both of time and distance. The example already given in which half a mile was walked in ten minutes is an unusually favourable one numerically. If a measured distance is used in an experiment it is wise to choose one which is easily converted to miles—for example, timing a marching group over 44 or 88 yards would be preferable to timing over 50 yards or 100 yards. Some qualitative appreciation that there may be variation of speed during the experiment and that it is an over-all or average measure that is found is taken for granted and noted rather than analysed. Treatment should be kept in the closest touch with everyday experience. Reading the dial of a speedometer is attractive to children with access to a car. The inverse relationship between speed and time is the fundamental difficulty to be overcome by experience.

Summary

It has been suggested throughout this chapter that the written reckoning of young children which is concerned with money, weights, and measures should be limited to reckoning for which they can see clear purpose. In many areas the teaching of elaborate ways of setting down compound rules still persists for the sake of examinations external to the school. Where this is so, it is suggested that the formal layout of such rules and their practice as a written skill should be postponed as long as possible. The more useful and relevant elements which go to make up the complicated procedures can be dealt with by mental and practical work with informal statement only. When at some later stage a more difficult case demanding several steps occurs or is introduced, it should first be dealt with experimentally. The formal layout deemed best can then be reached as a systematizing of something already there, instead of being taught as the way of doing yet another kind of process. When the form of procedure is agreed upon between teacher and child, it can appropriately be fixed as a habit of statement by suitable practice. Even with money, however, the time spent by young children on elaborate rules is to be regretted.

X

FRACTIONS

> Between the process of introducing a concept or idea into a child's mind and its bringing forth as an intelligible entity, which the conscious intellect is to handle and reason about, there should be a period of gestation, during which the idea is left dormant and passive in the child's mind; any attempt at premature interference is likely to result in abortion.
>
> MARY EVEREST BOOLE
>
> *Preparation of the Child for Science, 1904*

FRACTIONS may make a first appearance in the infant school, but they are not as a rule caught up into the computational machine until towards the end of primary school life or later. There is, therefore, a period of several years in which the development of the notion of a fraction takes place under environmental influences in which learning about fractions is more or less incidental. These circumstances are on the whole favourable to well-founded ideas about fractions, though even at a very early stage the outlook on fractions of the adults a child meets has a powerful effect. If the adult outlook is limited, so will be the adult forms of speech with which the child becomes familiar.

In Britain even the notation of vulgar fractions is learned early. An able child before coming to school may have noted how a halfpenny is written on a price ticket, or which packet is called a quarter of a pound of tea. The fractional form of speech is early associated with partition, but not necessarily at first with equal parts, and in the case of a quarter not necessarily with any particular number of parts. The half escapes this latter mistake because it is so simple and familiar; other fractional names are more clearly associated with number names than is the quarter.

Precision is often given to first ideas of fractions derived from common speech by the partition of some object or quantity to give fair shares. The cake may be cut up; the jug of lemonade shared; the length of tape folded to make team ribbons. The connexion between this partition into unit fractions and experiences preparatory to division is plain, but as a rule little is made of the possibilities of their common line of development.

As soon as measuring is a part of a child's activity, the partition of a unit measure to form a smaller unit arises. Fractional names for the smaller unit may be introduced. More than one of these smaller pieces call for something other than a unit fraction, for example, two-thirds of a pint and so on. It might even happen that 4 school milk bottles were referred to as four thirds of a pint, but 4 bottles is more likely. There are indeed very few instances in ordinary speech in which a fraction like $\frac{4}{3}$ is used, though the type $\frac{3}{4}$, $\frac{2}{3}$, &c., is quite common. Four-thirds in ordinary speech is generally used in comparing 4 units with 3 units of the same kind. For example, little Johnny is 3 feet high, Mary is 4 feet; Johnny's height is $\frac{3}{4}$ of Mary's, Mary's $\frac{4}{3}$ of Johnny's. Even in this example, unless there is a special reason for the second form, the first is preferred, and again the mixed form 'a third as much again' may be preferred to $\frac{4}{3}$. This bias amongst adults, due perhaps to their upbringing, has to be reckoned with in considering what the child is likely to learn from environment. Our methods of measuring, including our system of weights and measures and our system of coinage, favours the use of simple vulgar fractions; but it is usual to make statements about them in the form of a whole number of units and additional parts, $2\frac{1}{4}$ yards, $4\frac{3}{8}$ inches, and so on.

The early work supplementing everyday experience usually proceeds in several ways. Firstly there is that referred to above of giving precision to the idea of equal partition. This often includes experiments with cutting up different shapes or cutting a shape, such as a square, in half in a variety of ways. This kind of experience is closely allied to studying the nature of the shape, for the number of equal parts into which it can be cut by obvious means depends on its symmetry. Thus a triangle with two equal sides can clearly be cut in half, but it requires a good deal of knowledge beyond that of the primary stage to appreciate that it can be cut up into any number of equal areas by joining the vertex to points dividing the base into the appropriate number of parts. When the equality of parts has been emphasized, the independence of this equality of shape has to be illustrated. The difficulty here is that most diagrammatic approaches to fractions use areas in the representations of wholes and parts at a time when measurement of area has not been discussed, and the possibility of equal areas of different shape is still, if it is appreciated at all, an unexamined intuition. Different ways of dividing an area into two or more equal parts is as much a study of area as of fractions. Divisions by weight, for example, do not involve this particular difficulty. It is common to use a circular area as a convenient shape for the illustration of fractional parts, because of its perfect symmetry and divisibility. As a model for the illustration of fractions the partition of a length is, however, more powerful than the partition of a circular area, since a length can be immediately extended to cover the representation of improper fractions.

Another early supplement to normal experience is the more deliberate making of occasions for using fractional language and notation, already familiar in one or two instances, in a greater variety of contexts, in particular in describing relationships in less familiar measures, and later on in money. Increasing the number of contexts in which fractional notation is used may well be combined with the experimental approach to almost the only computational accomplishment in fractions appropriate to the primary stage, that is, the accomplishment which is the counterpart of a change of units, namely the principle of equivalence, $\frac{3}{4} = \frac{9}{12}$ and so on. If there is ample and varied opportunity for the description of what is being done when fractional notation is being used, it is almost certain that at some stage more than one right answer will appear, that is, someone will describe as $\frac{9}{12}$ or $\frac{6}{8}$ what someone else has described as $\frac{3}{4}$. This may be the right moment for giving practice in situations likely to involve a form of answer not in its lowest terms. Provision must also be made for the reverse process of converting a fraction to one with a larger denominator. These stages should be given time to become completely familiar, and a limited range of useful denominators should be used. Generally the eighths, tenths, and twelfths families, with occasional use of sixteenths and twenty-fourths and a few special cases, are ample.

Some examples are given below from a school which spread exercises involving fractional expression over the whole junior stage. There is nothing unusual in the approach, except that there was consistent association of apparatus and something done with verbal and symbolic expression at intervals over a long period. It was only in the last years that this went on side by side with simple fractional reckoning. The directions and questioning were sometimes unduly formal. Answers only are given below. Questions and apparatus must be imagined. The sample is a small random one taken from children's papers. The classes concerned were of about average ability, with a few children of good ability.

Age 7+

There are 2 quarters in a half. . . .
There are 2 eighths in a quarter.
There are 6 eighths in three quarters.
A $\frac{1}{2}$, a $\frac{1}{4}$, and a $\frac{1}{4}$ are a whole.
$1\frac{1}{2}$ inches must be added to $6\frac{1}{2}$ inches to make 8 inches.
$5\frac{1}{2}$ pieces can be cut from the red spill (2 inch pieces and an 11-inch spill).
The school bottle holds $\frac{1}{3}$ of a pint.
The cup holds $\frac{1}{3}$ of a pint.
There is $\frac{2}{3}$ of a pint left (after filling cup from 1 pint).

Age 8+

4 halves make 2 wholes.
8 halves make 4 wholes.
14 halves make 7 wholes.
8 quarters make 2 wholes.
16 ounces make 1 lb.

4 ounces make $\frac{1}{4}$ lb.
12 ounces make $\frac{3}{4}$ lb.
There are 60 minutes in 1 hour.
There are 30 minutes in $\frac{1}{2}$ hour.
There are 15 minutes in $\frac{1}{4}$ hour.
There are 50 minutes in $\frac{3}{4}$ hour (wrong).
There are 90 minutes in $1\frac{1}{2}$ hours.
There are 7 days in 1 week.
A day is $\frac{1}{7}$ of a week.

Age 9+

In the tin are several pieces of coloured cardboard.
They are fractions of the grey piece.
The green piece is a $\frac{1}{2}$ of the grey piece.
The fractions are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{12}$.
Two of the $\frac{1}{6}$ths make $\frac{1}{3}$.
The $\frac{1}{3}$ is bigger than the $\frac{1}{4}$.
Three $\frac{1}{12}$ make $\frac{1}{4}$.
If the grey piece stands for 60 mins., then the green = 30 mins., blue = 20 mins., red = 15 mins., yellow = 10 mins., orange = 5 mins.
12 parts in each inch and we call each part $\frac{1}{12}$ inch.
$\frac{3}{12}$ = one $\frac{1}{4}$.
$\frac{6}{12} = \frac{1}{2}$. $\frac{9}{12} = \frac{3}{4}$.
$\frac{15}{12} = 1\frac{1}{4}$ inches (mistake in statement).
2 inches = $\frac{24}{12}$ (mistake repeated).
There are four $1\frac{1}{2}$*d.* in sixpence.
There are four $1\frac{1}{2}$ yards in 6 yards.

The last two answers above are at this stage generally treated independently of one another, although they have the same number content. The first is familiar in experience, and experience is still a dominant influence in solutions. About this stage examples involving money became more frequent. In dealing with coins only particular fractions are appropriate, whereas with quantities such as length any fraction has meaning.

$\frac{3}{12}$ of a 1*s*. is 3*d*.
$\frac{8}{12}$ of a foot is 8 inches.
$\frac{1}{12}$ of £1 is 1*s*. 8*d*.
$\frac{1}{12}$ of 2*s*. 6*d*. is 2$\frac{1}{2}$*d*.
$\frac{3}{12}$ = 4*s*. 8*d*. (wrong).

There are 16 tablespoons in the medicine bottle (Drawing of bottle accompanies this with chalked quarters).
There would be 4 tablespoons in $\frac{1}{4}$ of the bottle.
There are 12 tablespoons in $\frac{3}{4}$ of the bottle.

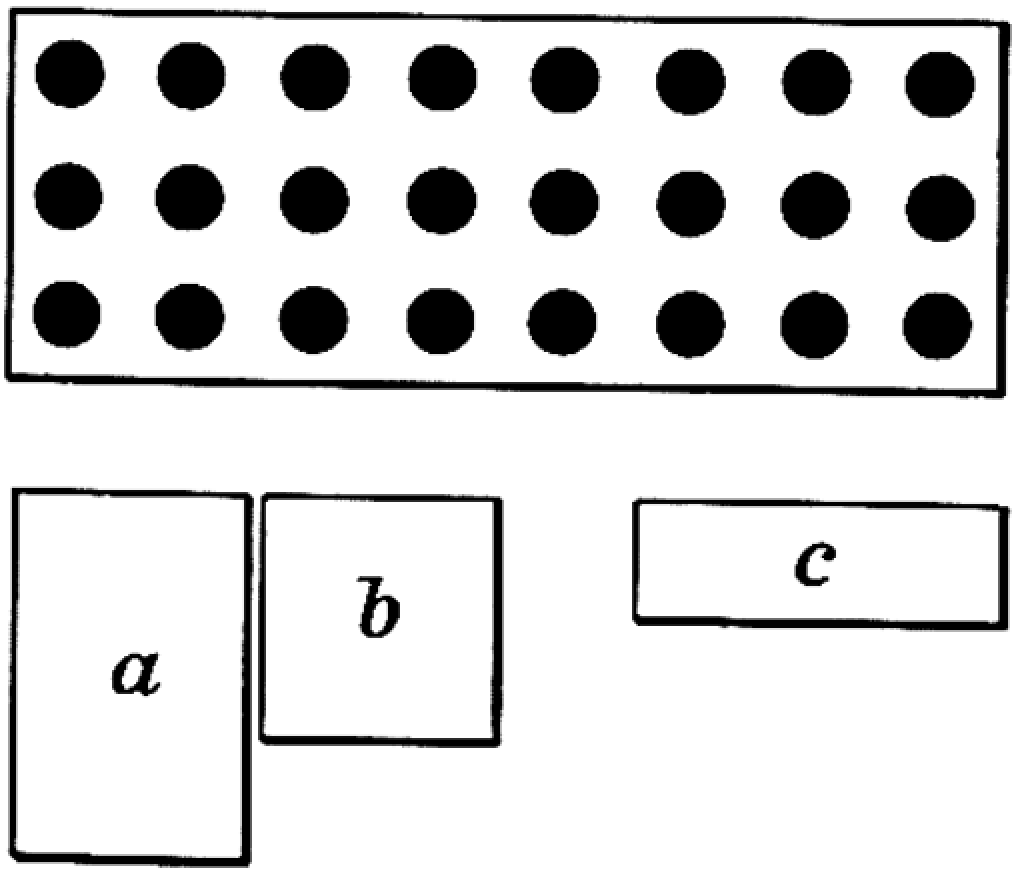


Fig. 36.

About this stage the equivalence rule emerges.

Josephine was a girl of practical rather than verbal ability. She was working with a card of twenty-four buttons (Fig. 36) and small cards, (*a*), (*b*), and (*c*), covering 6, 4, or 3 buttons. She explained quite clearly how card (*a*) covered 6 out of 24 buttons and how 4 such cards would cover them all. Her written account showed that she was gaining a very full appreciation of $\frac{6}{24} = \frac{1}{4}$, $\frac{4}{24} = \frac{1}{6}$, and $\frac{3}{24} = \frac{1}{8}$.

The fitting of card (*b*) is not quite straightforward.

> Daisy, a child at about the same stage, was working on twelfths of an inch. All she did at first required reduction to larger units, for example, $\frac{9}{12}$ inch = $\frac{3}{4}$ inch. She was unable to pass straight from this to an equivalent for $\frac{5}{6}$ inch, but she had no difficulty with $\frac{1}{6}$ inch = $\frac{2}{12}$ inch. From this she delightedly obtained the $\frac{10}{12}$ inch.

From the stage at which children are prepared to convert one fraction to an equivalent one of different denomination they are in a good position for dealing with any practical situation on its merits. It is common practice to hurry on from this stage to the teaching of rules for addition, subtraction, multiplication, and division of fractions. The last two at least of these would be better left until fractions are more firmly settled into the family of numbers. This does not mean that situations involving what may be called 'of-ing' or even 'by-ing' cannot be solved from first principles. Addition and subtraction involve so little difficulty that they generally appear spontaneously. In the examples given, forms of speech associated with addition appear amongst the answers in the 7+ age group. It is a pity to formalize procedures into a rule, as common denominators for a number of fractions will be selected by inspection if families of fractions involved are in a familiar range. The possible difficulty associated with mixed numbers and a difference, for example $2\frac{1}{3}-1\frac{1}{2}$, will not be insuperable if preparatory work such as that shown in the 8+ group's answers has been understood. It is a pity to use the long line over a single denominator, for example, to write

$$\frac{6}{6}+\frac{2}{6}-\frac{3}{6} \quad \text{as} \quad \frac{6+2-3}{6}.$$

An instance of treatment by first principles of a situation which might at a later stage be treated by a rule of multiplication is given opposite in some detail. Although the question was tried with many children of 10 and 11 years of age who had learned a multiplication rule, when the question was presented in a setting which suggested a *practical* approach, only one child of exceptional ability obtained the answer by a purely numerical method.

> *Question.* In making copies of a number of national flags Jane had to copy one with three stripes of equal width. Her flag had to be $3\frac{1}{2}$ inches across. How wide was each stripe?
>
> Letitia, a very able girl of 10, immediately said $1\frac{1}{6}$ inches having no doubt taken $\frac{1}{3}$ of $\frac{1}{2}$ without stopping to think. No other child of her age among many answered like this.
>
> Mary, John, and many others found first that it was more than 1 inch, and then changed the $\frac{1}{2}$ inch to $\frac{3}{6}$ inch or $\frac{6}{12}$ inch and shared by 3.
>
> Janet, who was 9 years old only and not very clever, but who liked doing simple scale drawing, said 'If the $\frac{1}{2}$ inch was a $\frac{1}{2}$ foot, $\frac{1}{3}$ of this would be 2 inches, and 2 inches is $\frac{2}{12}$ feet, so I want $\frac{2}{12}$ inches'. Other children have been found who reach the answer this way.
>
> Conway, aged 11, in a secondary modern school, preferred halving to taking $\frac{1}{3}$ of, so he said 'If the $\frac{1}{2}$ inch had been a whole inch, $\frac{1}{3}$ of it would be $\frac{1}{3}$ inch; but I only had $\frac{1}{2}$ inch, so I want $\frac{1}{2}$ of $\frac{1}{3}$ inch which is $\frac{1}{6}$ inch.' (This is not an uncommon method.)
>
> Bill, an older boy, and others familiar with working in eighths in the woodwork room, made incorrect estimates in eighths.

The most common method is Mary's and John's, which is a direct use of the important principle of equivalence. Opportunity for using this principle in tackling problems, especially practical problems, is appropriate at the junior stage while the behaviour of fractions in a variety of situations is becoming familiar. Conway's method is an example of preference for a familiar number. Two is, of course, a number for which there is a strong preference from an early age, and three is very much less familiar in practice.

Up to the stage of appreciation of the principle of equivalence, and with it the implication that a fraction is a relationship between numbers, a line of advance such as the one sketched out can be kept in close association with British children's experience. That this is so is partly due to our system of money, weights, and measures, and partly to adult habits of mind acquired from what has become a traditional approach. Able children can certainly reach with understanding the stage outlined by the age of 11 and children of average ability will not lose their way if the approach is careful and not too intensive. If this line of advance is followed, it is suggested that even for the ablest there should be pause for the principle of equivalence and its implications to sink in, and for an experimental use of it to give it the emphasis it requires. The next stage of treating the fraction relationship as an entity to which the word number can conveniently be extended, along with meanings for addition, subtraction, multiplication, and division, is one which can only be made very slowly if difficulties are not to be created for able minds and confusion for the less able. Rules can, of course, be learned, but as shown above they will not be used with understanding. The acceptance of a fraction as a number is just taking root and must develop undisturbed.

> Two sisters were asked 'How many numbers are there between 3 and 7?' The younger unhesitatingly gave the number of intervening integers, the elder included intervening quarters, halves, and three-quarters. The elder was prepared after a little stimulation to extend the number of 'numbers' indefinitely to include all fractions.

Such a growth should not be forced and should certainly not be made explicit at the primary stage. It is probably helped more than anything by allowing fractions to take their place on a linear scale in which fractional parts of a length fall in between whole units. All sorts of drawing and measuring activities are, therefore, most helpful.

For children able enough to reach the principle of equivalence at the primary stage, there is some danger that the notation *a* over *b* for fractions may become so strongly associated with fractional contexts that it cannot be easily transferred to other situations,

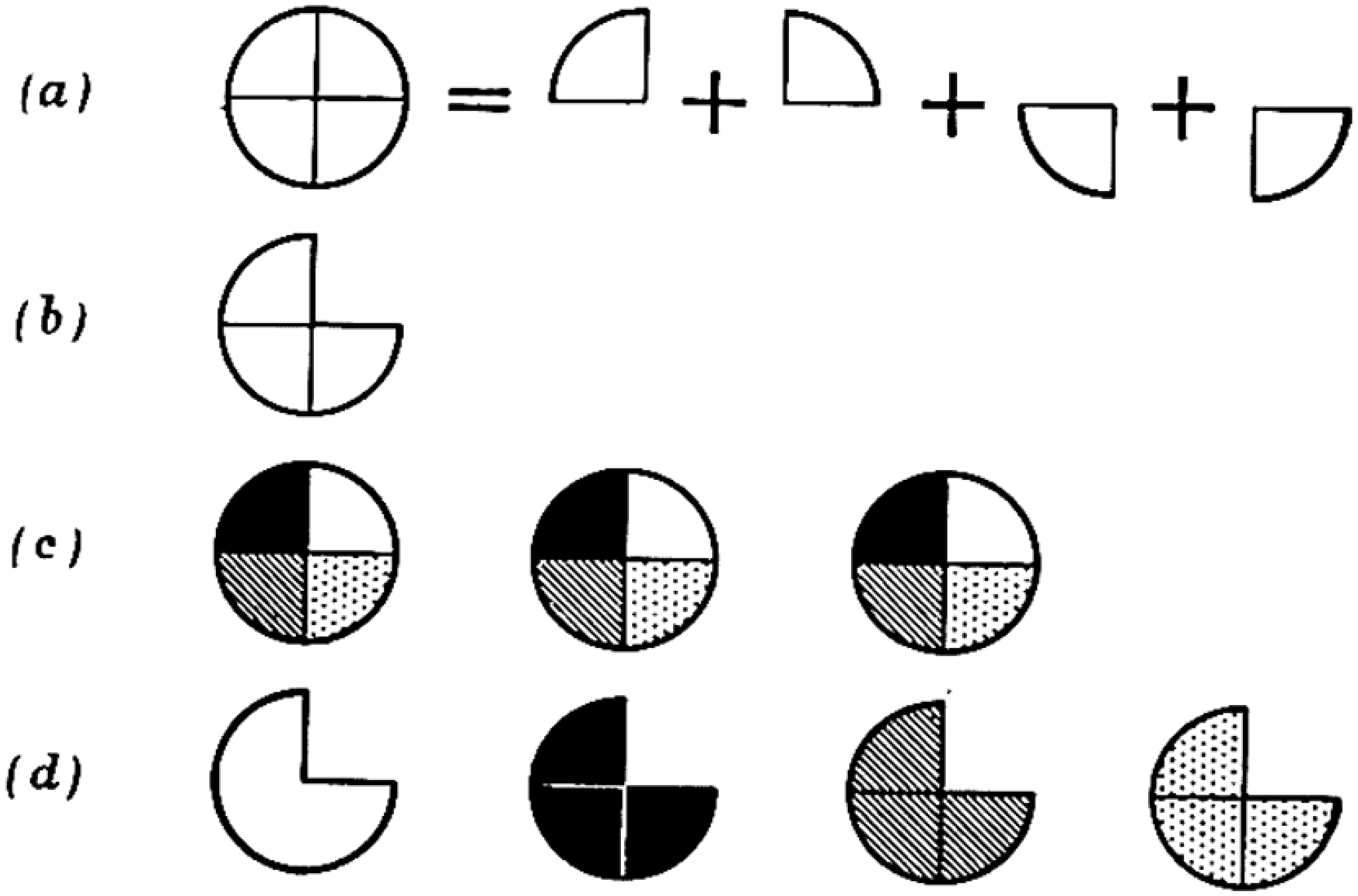


FIG. 37.

such as sharing and measuring, which yield the same number relationship. The ratio of parts taken to parts in a whole, which is involved in a fraction, is not the only situation in which ratio occurs and in which therefore the notation *a* over *b* is appropriate. This 'bar' notation, that is, for example, three-quarters written 3 over bar 4, arises in Britain from environment in fractional contexts, particularly from the writing of $\frac{1}{2}$*d*., $\frac{1}{4}$*d*., and $\frac{3}{4}$*d*., for fractions of a penny. It has been stated earlier that so long as the unit fraction only is concerned this is readily associated with sharing; one whole is cut up into 4 parts in sharing between 4 people and each part is written $\frac{1}{4}$ (Fig. 37 *a*). When 3 of these parts are taken, the situation involves taking three quarters of one whole (*b*) and the fraction is written $\frac{3}{4}$. This is quite different from the situation when children start with 3 wholes, divide them all into quarters (*c*) and then take one quarter from each whole to make up four equal shares (*d*). These shares are equivalent to three-quarters of one whole and would be written $\frac{3}{4}$ but they are built up in quite a different way from (*b*). The same difference might be illustrated in finding, by the process of folding strips of paper, $\frac{3}{4}$ of a yard, or $\frac{1}{4}$ of 3 yards. These are, however, uncommon cases of sharing, at least uncommon in the arithmetic books; in practice, of course, Father may come home unexpectedly to tea so that the 3 buns must serve for a family of four instead of three. Arithmetically it is more usual to share some number of units which is greater than the number of shares, such as 8 wholes or 9 wholes to be shared among four people; $\frac{8}{4}$ or $\frac{9}{4}$ are, however, uncommon in experience of fractions. It is suggested that if experiences of sharing and experiences of fractions were not kept unnaturally apart by the use of different notations, the bar notation might be enriched and the nature of the relationship made clearer while dealing with very simple numbers, especially such 'whole number' cases as $\frac{8}{4} = \frac{2}{1}$.

An example is given earlier in this chapter of a ratio which is not a ratio of a part to a whole, but the comparison by quotient of two quantities of the same kind, namely Johnny's height of 3 feet with Mary's height of 4 feet. This can be given as Johnny's height is $\frac{3}{4}$ of Mary's or Mary's height is $\frac{4}{3}$ of Johnny's. Supposing Johnny's Father is 6 feet tall, then Johnny's height is $\frac{3}{6}$ of Father's and Father's height is $\frac{6}{3}$ of Johnny's. The $\frac{6}{3} = \frac{2}{1}$ might arise from the question 'How many pieces 3 feet long can you cut from a piece 6 feet long?', showing up clearly the underlying unity between the situation giving rise to a ratio and what is called a 'measuring' situation giving rise to division. If we could be content with small numbers until such simple cases as the (6, 3, 2)-relationship of multiplication-division were established in its many incarnations, it might be possible to use both the bar notation and $\div$ without confusion. On the whole the bar notation seems preferable.

The example of ratio of heights is an easy instance of a much neglected use of the bar notation. A rich experience may easily produce examples in which this form of comparison is much more appropriate than comparison by difference and in which the bar notation can be used. The likeness of comparison of one thing with another to comparison of a part with a whole is so strong that it is not confusing, though it is not immediately obvious in the less common improper fraction form. For example, Johnny's pocket money may be $\frac{3}{4}$ of Mary's in the sense of comparison of one thing with another, but it will be a very small fraction of Father's weekly wage, of which it is actually a part.

It is difficult to determine how far neglect of opportunity is due to the set of adult ideas and how far it is caused by pressure to elaborate processes, such as division, whether or not there are ideas behind these processes. Pupils who have enough ability to do correctly the complicated computation often taught to juniors might be able at least equally well to use this powerful bar notation in a descriptive way in a variety of contexts without confusion, so long as numbers were kept small. Children of lesser ability need all the longer to make their adjustments. Many of them who follow courses of computation have little idea of what division is about and clear notions of only a very few particular fractions, such as quarters, which are of such frequent occurrence that they can hardly be missed.

Decimal fractions on the other hand have in Britain no such easy entry through experience at the primary school age. Most of the commoner usages of the notation are specialized and more likely to fit in with later interests. Probably the best early introduction to the use of the notation is its use in drawing and measuring in inches and tenths of an inch. The notation for tenths can then be used at the same time as it is shown to be an extension of place value. Facility in speaking of a length in the alternative forms 2·7 inches and 27 tenths of an inch marks mastery of this stage if it is something more than mere application of a rule. The estimation in 'tenths-of-a-tenth' of a length, which is not closely in agreement with the tenths marking on a ruler or squared paper, is the next stage. The occurrence of such opportunities depends on the content of the geometrical work; the estimation of a part of an interval in tenths, or in any other fractional part, is a judgement of slow growth and great importance. Generally the first stage is estimation of more or less than a half, that is, more or less than five tenths. To determine the second decimal place as tenths-of-a-tenth is more informative than to define it as hundredths of a unit.

The usual plan of defining the decimal notation and showing it to be an extension of the units, tens, and hundreds plan for writing numbers is a theoretical approach which fits in with computation but not with significant use. It does not even lead to the kind of simple usage described above. This failure has been tested many times with older children who have learned to compute with decimals. The following story is of a typical reaction.

> A class of age 11 and 12 years of just above average ability were quite efficiently working the four rules with decimals. They were asked to show a length of 1·8″ on their rulers. Most but not all of them chose the tenths and reported that they put a pointer on the eighth mark after the first inch. They were then asked to show whereabouts to put a pointer for 2·37″. No child in the class was able to do this without help, and some put a pointer on 3″. With questioning they were able to place the length between 2·30″ and 2·40″ but nearer to the 2·40″ mark. They then answered questions about lengths still nearer to 2·4″ and produced 2·38″, 2·39″, 2·395″, 2·399″ &c.
>
> The 13-year-old class of corresponding ability were using decimal methods of calculation in dealing with the rates of the locality. When asked what use ·07*d.* was in the data given for determining a rate, a girl immediately replied 'If a lot of people paid it, it would be worth something'.

Although the correct reply would have been 'If the rateable value was high . . .', she had the essential idea of increase by multiplication.

> When the class was asked to show 2·37″ in the same way as the younger class, they had exactly the same difficulty.

There is, indeed, so little use of a purely descriptive kind for decimals at the age when the notation is frequently taught that this essential notion of arrangement in order of magnitude is missed, and decimals are merely developed in relation to computation. The power of this notation in computation is one of its greatest assets, but it is more suited to the secondary than the primary stage. So, too, is the descriptive use of the notation associated with large numbers. To say that 3,840,000 is 3·84 millions creates a million as a new unit. But there are few contexts suitable for primary pupils in which numbers of this kind would appear. Similarly foreign measures and money are usually left to a later age.

The ordering of decimals given to two places is just within primary usages in connexion with measuring and drawing. The use of decimally numbered labels in a library is an excellent example of this ordering; decimal labels define an order with the possibility of inserting a new book between any two books without upsetting labels already given. Thus 3·1 and 3·2 may include 3·11, 3·12, 3·13, 3·14, &c., and a third place of decimals would allow additional insertions.

It is usual to teach the decimals which correspond to the commonest vulgar fractions, such as 0·5, 0·25, and 0·75. A few such landmarks for comparison are convenient, but the usages of vulgar fractions and decimals seldom overlap; decimals are appropriate as a rule in circumstances requiring order of magnitude or approximation, whereas vulgar fractions arise in circumstances of precision.

Few occasions arise for using decimal calculations. Adding and subtracting are not difficult, but neither are they very likely to be needed. Multiplication or division of a decimal by an integer, if it occurs, can be treated by common sense. Multiplication and division of decimals by decimals are generally taught in advance of usage. The kind of example which might occur is the finding of a distance on a map with a scale of 2 inches to 5 miles. A measured length on the map of 7·35 inches might be converted to miles by using the equivalent 1 inch to $\frac{2}{5}$ mile or 0·4 mile. The experimental building up of a result for this using either $\frac{2}{5}$ or 0·4 would be an interesting problem for more able children. It would be sanctioned at each stage by its relation to distances more easily calculated. There are very few such examples likely to fall within the experience of even an able child of eleven.

The expression of a vulgar fraction as a decimal is a particular case of division by an integer, which sometimes creates difficulty because the bar notation has been too exclusively associated with a fractional part. As a general process this is not recommended in the primary school; particular cases are amenable to other means and can be found experimentally.

The form of a fraction which we call a percentage has been sanctioned by common usage and cannot, therefore, be ignored. Historically per cent. and per thousand preceded decimals and served purposes for which decimals might now quite well be used. The process of expressing a fraction as a percentage is just another example of change of denominator by using the principle of equivalence. The only difficulty in this change is dealing with fractions not in the 'hundredths family'; thus twentieths are readily converted to hundredths, but thirtieths involve a fractional percentage because the factor 3 in 30 is not a factor of 100.

The use of percentages descriptively can begin quite early with such simple cases as 50 per cent., 25 per cent., and 75 per cent. Also 99 per cent. as 'very nearly all' and 100 per cent. can be appreciated in contexts such as ranking attendances in order of merit, which are fairly well understood by a child. He accepts the percentage given by the teacher as a measure in much the same way as the infant boy enthusiast for speed says 'The car was going a hundred!' Rather later the percentage, like the decimal, gains from the practical importance of ranging fractions in order of magnitude in certain situations. For example, attendances in different classes or marks with different maxima are to be compared. The child has first to reach the stage of recognizing the ratio of score obtained to possible score as a measure of success. This intuitive appreciation cannot be forced, though it may be clarified by examples which are easy numerically, for example, by comparing scores on maxima of 20, 50, and 100. Thus 19 out of 20 is not as good as 48 out of 50, if circumstances make this ratio method a fair one.

Fig. 38 shows some diagrams and percentage attendance figures extracted from the daily school records and summary by an able girl, Letitia, mentioned earlier in this chapter, who was working on individual graphical projects in advance of most of her fellows.

Pres.	*192*	*200*	*204*	*170*	*200*	*966*
Max.	240	240	240	240	240	1,200
%	80%	83·3%	85%	70·8%	83·3%	80·5%

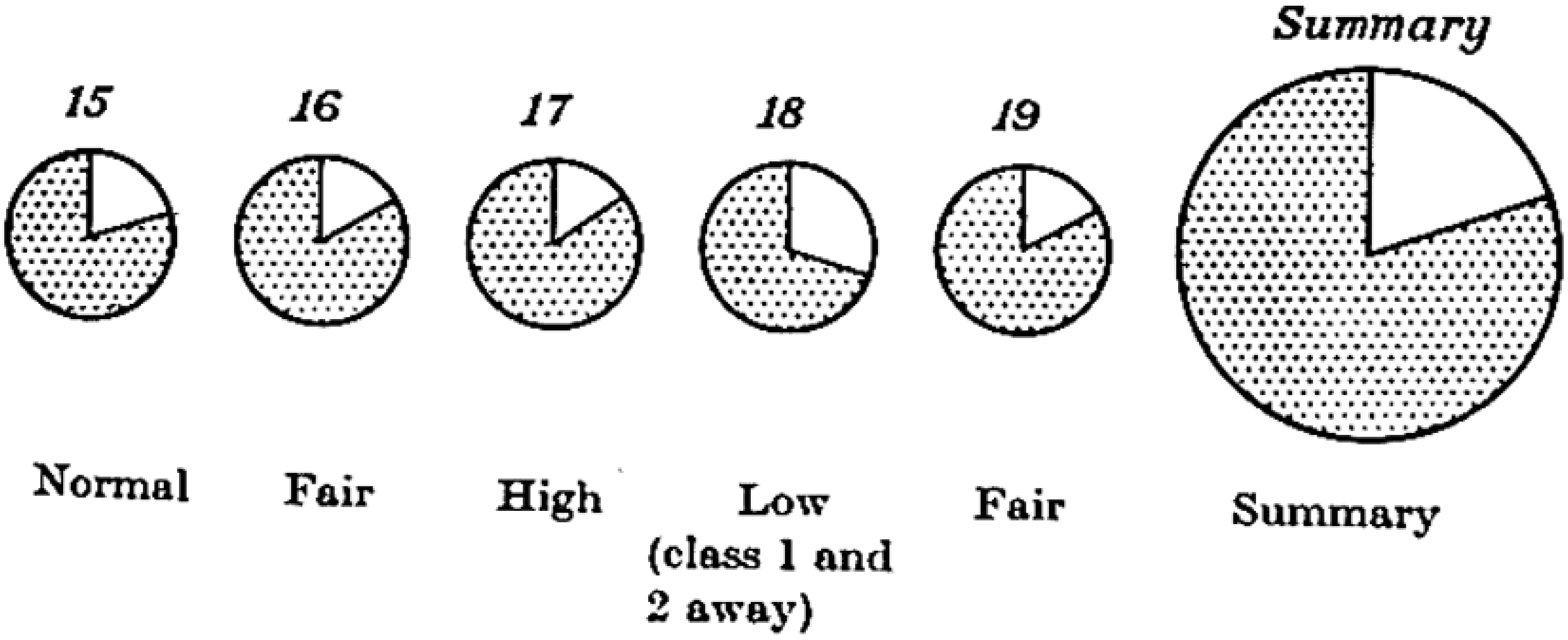


Fig. 38.

The angular measures were worked out so that in each case the shaded portion was the correct fraction of the whole circle. The summary circle was drawn so that its area was equal to five of the small circles, but it was not made clear in the diagram what knowledge Letitia brought to this more advanced part of the construction.

XI

DRAWING, MODEL MAKING, DIAGRAMS

THERE is a reference to the development of geometrical vocabulary and ideas at the end of Chapter III, and some of the numerical aspects of measuring lengths, areas, and volumes have been discussed in the two preceding chapters. In this chapter the threads of progress will be gathered together and consideration given to the range of opportunities for primary school children to advance in their geometrical ideas and techniques through making, through drawing, and through representing data diagrammatically. These opportunities will be considered for their own sakes as interesting activities and for their contribution to the development of mathematical concepts.

Ruler drawing for boys was at one time a firmly established part of the junior class programme; it generally offset the sewing lessons of the girls, and it was looked upon as a preparation for work in the woodwork room after the age of eleven. As mere technique and procedure it was frequently overdone; but the experiences it involved of shape and of number applied to measuring lengths were sadly missed by the girls, whose arithmetic held little but figuring. Although the reasons advanced for this differentiation in the mathematics of junior boys and girls are falling into disrepute, the fashion for giving boys greater opportunity for the development of geometrical ideas does still persist in many schools; courses for boys only still exceed the additional practice in draughtsmanship which might justifiably be given to boys just before transfer to a secondary school. The courses themselves, however, whether for boys only or for boys and girls, are becoming more closely linked to children's interests and constructive purposes; they are less dominated than formerly by a set succession of techniques. For example, expression work of a constructive kind often produces a purposeful effort which can be harnessed to achieve incidentally some of the objectives of the old formal drawing courses.

Another formal course which has long been discarded in most schools is paper folding as a handwork exercise. It is a pity that with its passing has gone much mathematical experience involved in the folding of a piece of paper. For example, tearing scraps from a paper folded twice at right angles, so as to make a patterned mat like a cut paper doily, is a good experience of symmetry. The number of axes of symmetry can be increased by increasing the number of folds.

The double folding of a scrap of paper, so that the first fold falls on itself (Fig. 39) is indeed a valuable geometrical instrument useful for informal making and checking of right angles and showing in its construction the relation between 'a square corner' and a straight edge.

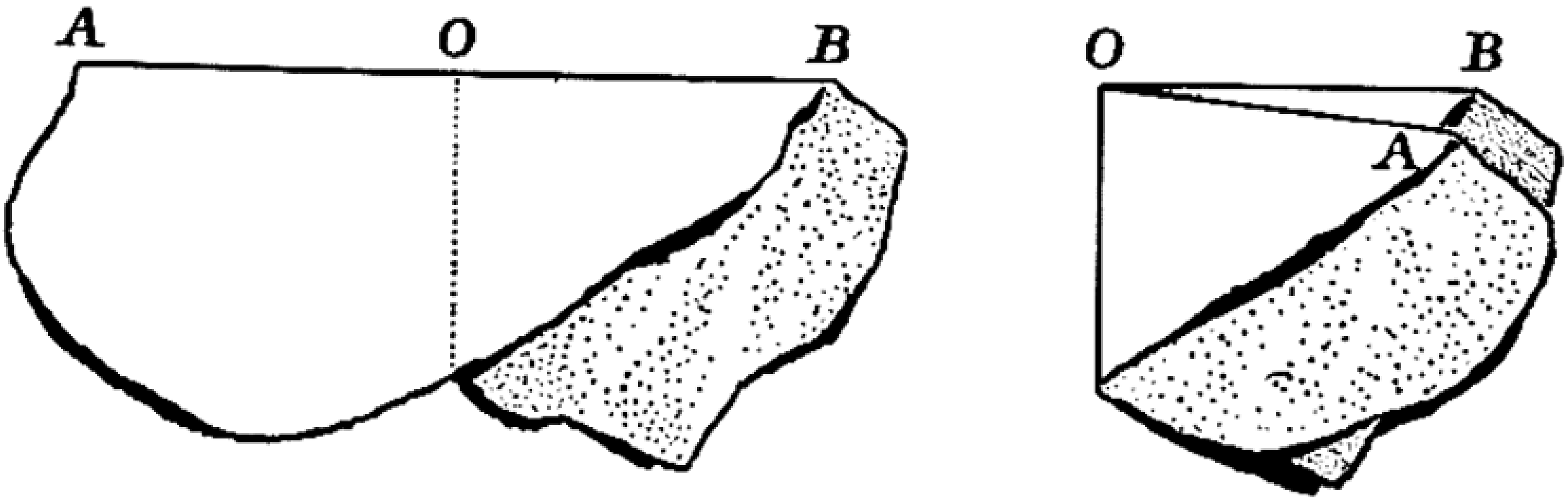


FIG. 39.

Again if sub-divisions of a line be needed, for example for drawing a pattern, a rudimentary way of obtaining sub-division is to fold an equivalent length of paper for use as marker. This is easy if the required number of sections is a power of two (2, 4, 8, &c.), but requires trial and error for, say, 6 parts, and is extremely difficult to achieve with reasonable accuracy by trial for 5 parts. For 15 parts some other method is called for if the whole line is to be used. The stepping off of a chosen length for a pattern unit, with the possibility of an unused section of the given strip, is a fundamental experience of the aspect of division called measuring, whereas the folding of the paper strip is an experience of sharing.

Folding is also a most informative process when applied to the construction or examination of shapes. As a means of producing symmetry it is unrivalled. The old game with a scrap of paper and a damp signature folded over to produce a 'ghost' is a quite serious geometrical experience; the axis of symmetry is clearly marked by the fold, and it has a looking-glass quality not easily obtained by other means (Fig. 40).

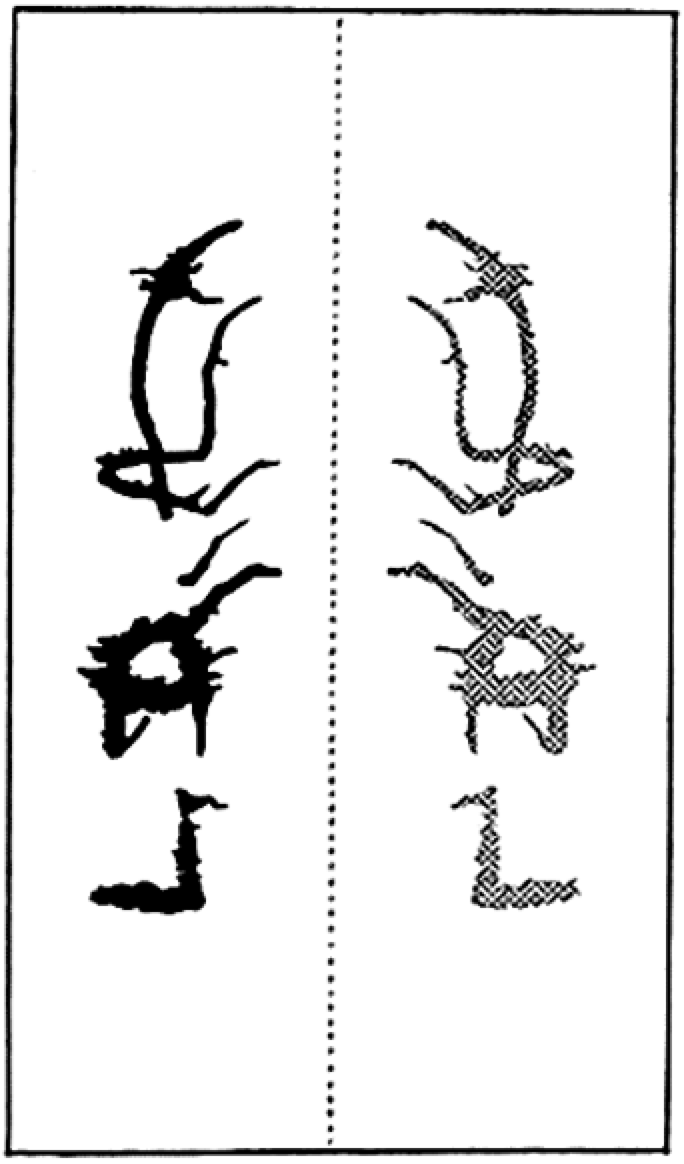


FIG. 40.

Folding is particularly informative if a shape has point symmetry. For example, to crease in the diagonals of a square and a rectangle (though it is not easy to do well) shows more than drawing them in. Such folding might occur in trying to cut figures in half in a variety of ways, and it would bring out most vividly that the halves of the rectangle had to be turned over before they would fit in shape as well as be the same size. It would be better still if this folding occurred in the course of some non-mathematical pursuit, because this kind of observation is best made when attention is directed to its practical consequences rather than to mathematical information for its own sake. The inquiring child who asks if the diagonals of a rectangle cut the rectangle in quarters has set himself a nice exercise in cutting and fitting (Fig. 41) to show that the two pairs of triangles can be combined to form exactly equal shapes. Such an exercise is suitable at the primary stage if it is spontaneous. It is in the nature of a puzzle.

Model making with paper and cardboard, though no longer common as a graded course of handwork, is still a favoured activity from quite an early age. The making of a 'tent' from paper

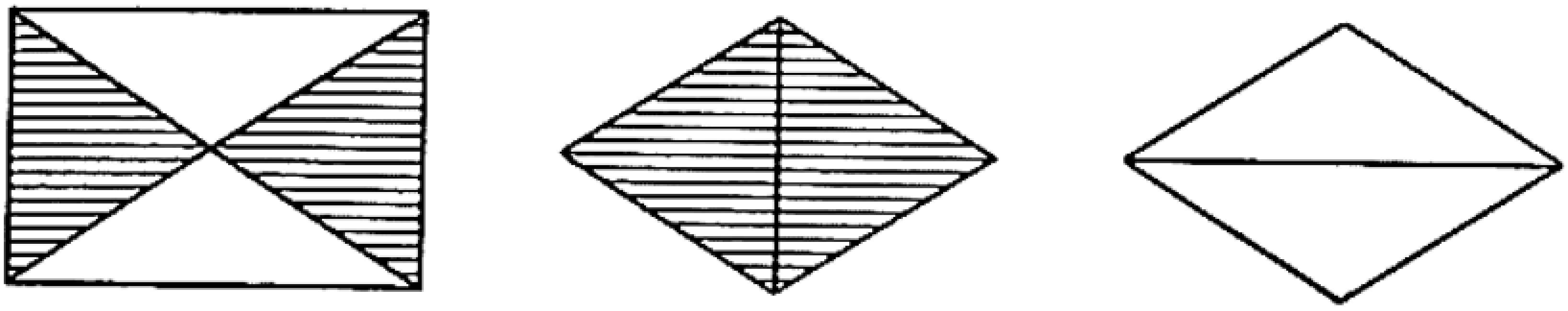
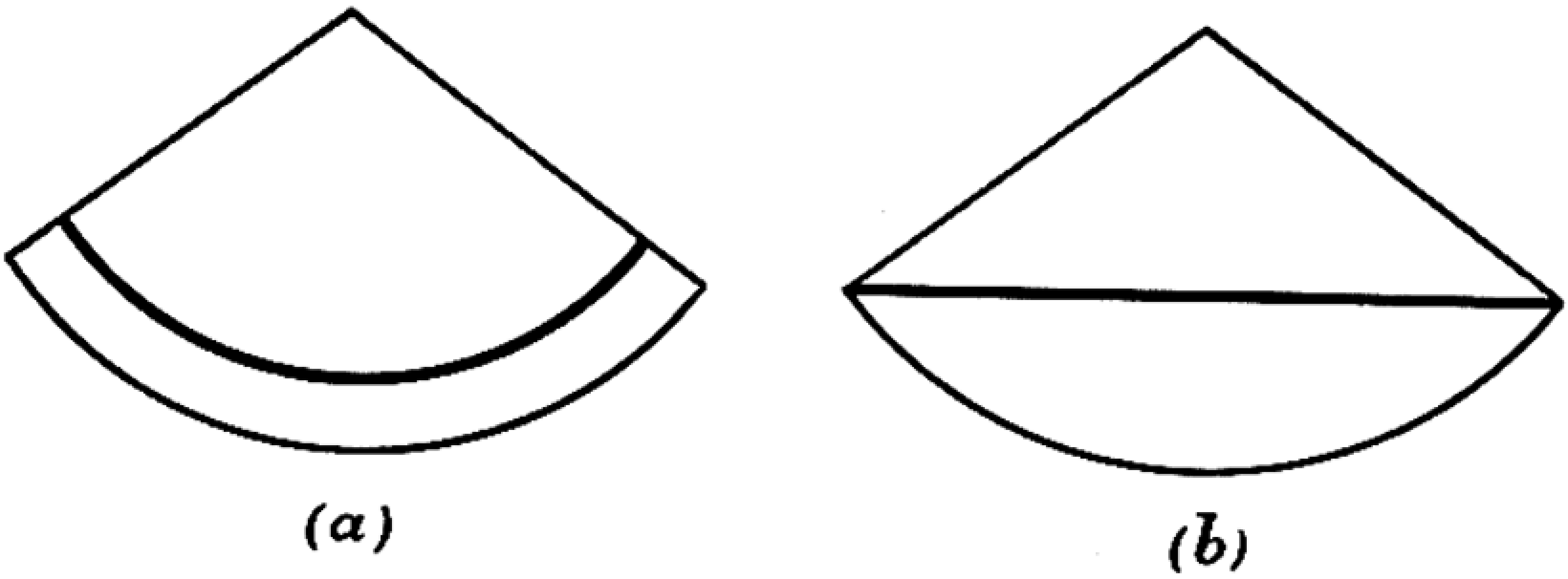


FIG. 41.

is an experience which should not be spoiled by instructions in how it should be done. Many first experiments will result in failure. Folding a triangle clearly does not make a conical tent that looks satisfactory. Cutting down the folded paper till it stands better may produce an approximation to part of a circle. Giving this precision, and determining how much of the circle should be used to give a shapely tent, may be a question for help and advice or for experiment and discussion according to the previous knowledge about circles and angles which the child possesses. A group of young juniors may find in this making of tents a first opportunity for using a mechanical means of drawing a shape, the circle, already familiar in experience. The pin and loop method is to be preferred at early stages to the use of compasses. Decorating the tent sometimes produces an interesting point for discussion; it is quite common to find a child trying to put a border pattern round the base as in Fig. 42 *a* but putting in a chord instead of a concentric circle as in Fig. 42 *b* so reproducing



(a) *(b)*

FIG. 42.

the original mistake of the triangle. This most commonly occurs when children have been told, rather than allowed to discover, how to cut the paper for the tent, or have even been given ready cut circular sectors.

At later stages some modelling of solid forms may be undertaken for the sake of geometrical experience, but more generally this work will be incidental to the construction of something the child wants for a purpose. As much trial and error as possible should be allowed in determining the pattern or net which has to be cut out and folded. To see that a tray with sides 1 inch high at right angles to the base can be made from one piece of paper consisting of a base with rectangles 1 inch wide attached to each side (Fig. 43) is more important than the actual construction. It can be given a lid as a next stage. Those who find it difficult can dissect something already made, and the child who cuts all the pieces separately to begin with, only has to join them up again later. The little houses which are a paper shell are quite an advanced example of the casing of a solid made from folding a flat 'net'. If the house be made all in one piece, considerable ingenuity is involved. The matching of lengths and folding along intersections of planes is an important geometrical experience.

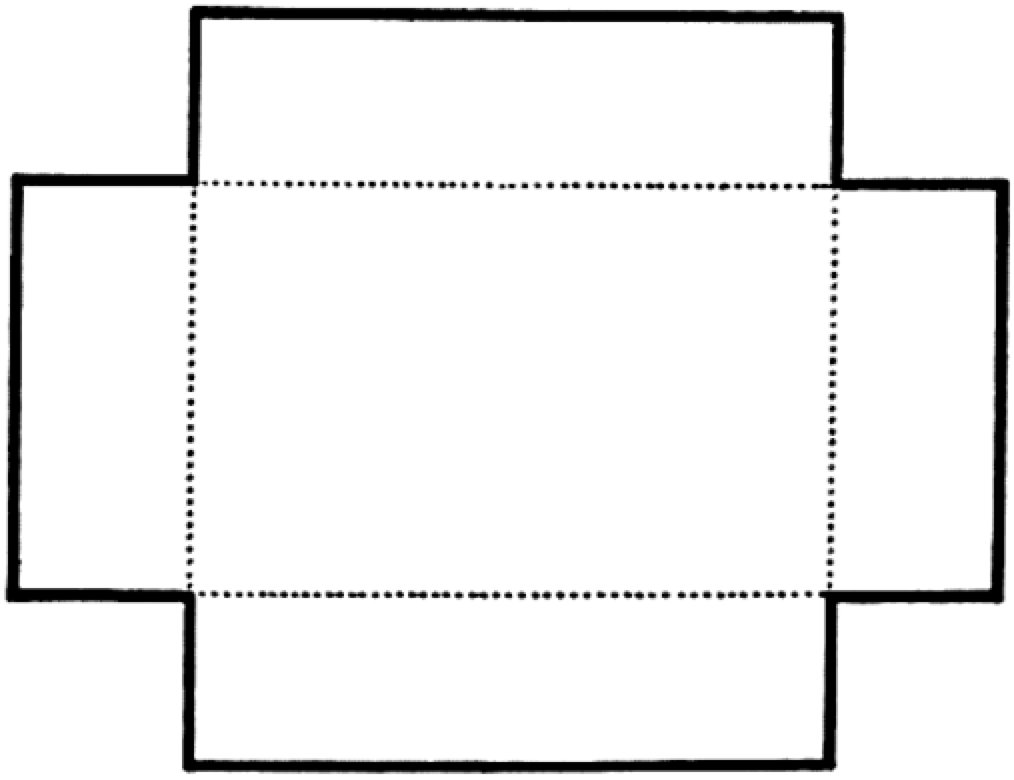


FIG. 43.

Any geometrical experience which involves drawing and making will need simple geometrical tools. Primitive tools and measurements gradually give way to more refinements of draughtsmanship; but it is often at the primitive stage that ideas are clarified, for example, the pin and loop description of a circle, or the corresponding stake and cord out of doors, shows the essential property of a circle in relation to its centre more clearly than compasses.

The turning of a top with a circle marked round it is a demonstration of the circle's perfect symmetry; so is the use of the potter's wheel, or the way a bird turns to shape its nest.

Movement has a fascination for children, so that any approach to geometrical ideas involving turning or moving along is at least as important as a static observation of shapes, directions, and positions. Noticing the angular properties of shapes, that is, 'more and less pointed', 'not a square corner, but sharper', may, however, go on side by side with the appreciation of the effects of turning, that is, 'I turned right round or half-way round', 'turn and look out of the window'. A culmination of the appreciation that turning can be more or less is its association with the turning of some line, for example, the line of sight, from a first to a final direction, and the understanding use of the phrase 'turning through an angle'. The bodily movements associated with turns in a horizontal or vertical plane give angles so situated an early claim on attention. On the whole the association of the first measuring of angles with a turning movement is more vivid than the static comparison with, for example, a piece of paper folded twice, as in Fig. 40, to give a right angle and then again to give a finer scale of comparison. The second can be used in association with the first as a convenient way of obtaining a representation of a compass dial when turning has been associated with turns from the direction facing north. There are a great number of really practical and outdoor examples associated with direction and angles, some at least of which should be explored before simple angular measurement is reduced to drawings made at a desk.

Most of this kind of geometrical drawing is best left to the secondary stage.

> Some 7-year-old children passed from an interest in a picture of flying geese by Peter Scott to a discussion of Captain Scott, whose explorations some of them had recently seen in a film. A boy asked 'How did they find their way over the ice?' Compass needles were produced and floated in saucers of water.

The children made a memorable first step in the discussion of direction, from which noting compass points in their playground and half and quarter turns would readily follow. Several magnets all turning to face the same way also provided a more vivid example of parallels than the static lines of rooms and furniture. Direction independent of position is not an easy idea; it is often blurred by the comparison of directions being limited to the direction of lines drawn from a single point.

> Another group of 8- and 9-year-old children became extremely interested in shadows. The changing length of a shadow attracted them first, and they gained a rather tentative feeling for the angular elevation of the sun. This was clarified by a little height-finding, using an equal-sided right-angled triangle held with a side horizontal. They readily appreciated that an object sighted along the hypotenuse was as much above their eye as it was distant from their viewpoint.

This gave them a clear picture of the shadow which was the same length as the object casting it. Angles of elevation are not difficult to deal with practically by simple means with children of only moderate ability. This particular group of children carried their interest in shadows into other fields. They observed directions of shadows at different times of day, as well as shadow lengths, and they were never tired of observing differences in the shapes of shadows as objects changed their position or shadows were cast on different surfaces.

Among ideas which can be given a practical precision before the end of the primary stage is that of the vertical line or plane by the use of a plumb-bob and line, in connexion, for example, with the use of simple height-finding apparatus or in some piece of constructive hand-work which should stand upright. The horizontal can be related to the vertical. The spirit level is a rather harder instrument than the plumb-line. To generate a horizontal plane by revolving an arm set out at right angles from a vertical axis of revolution would be the kind of experience of motion likely to make an appeal. Any continuous change resulting in the evolution of a line, surface, or solid is an appealing geometrical experience. The close observation which quite young children give to moving parts of machinery from a simple pendulum to elaborately co-ordinated big machines is seldom exploited in schools.

> Frances, aged 6½, watching a set of model steam-engines, remarked 'I like to see the up and down go round and round'.

A separation of the idea of size from the idea of shape requires an adaptation to the experience that objects appear to grow larger as we approach them. This is a first primitive step towards appreciating drawings to scale. Some estimation of the relative size of objects may appear very young.

> James, a 4-year-old, named three large streaks and one small streak of painting 'Cows and calf' (Fig. 44).

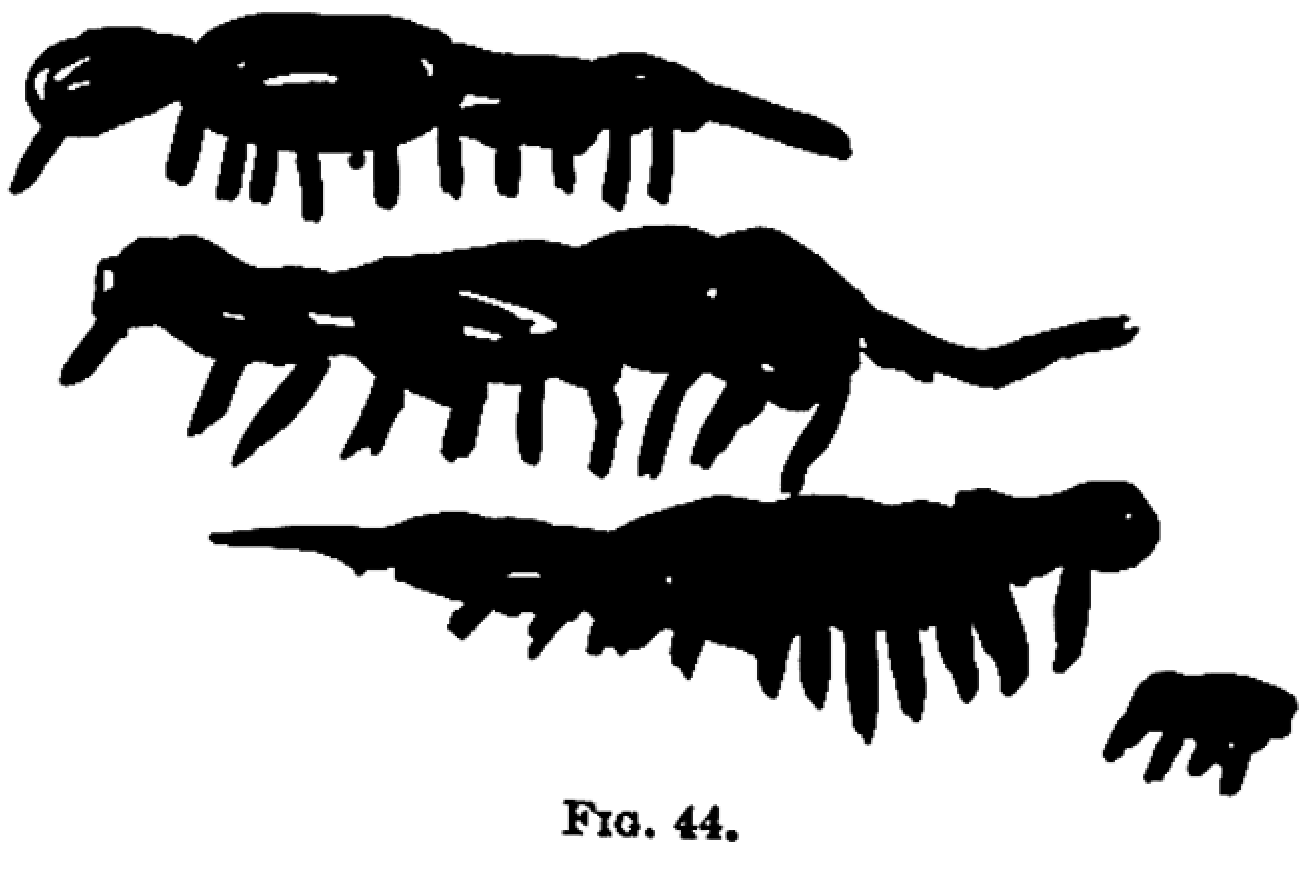


FIG. 44.

Although the number of 'legs' shown was in excess of four, the relative sizes of cow and calf were suitable.

Bill, on the other hand, aged 6, drew 'what he did on a day's holiday', as in Fig. 45, and when asked what it was, he said 'Lollipop'.

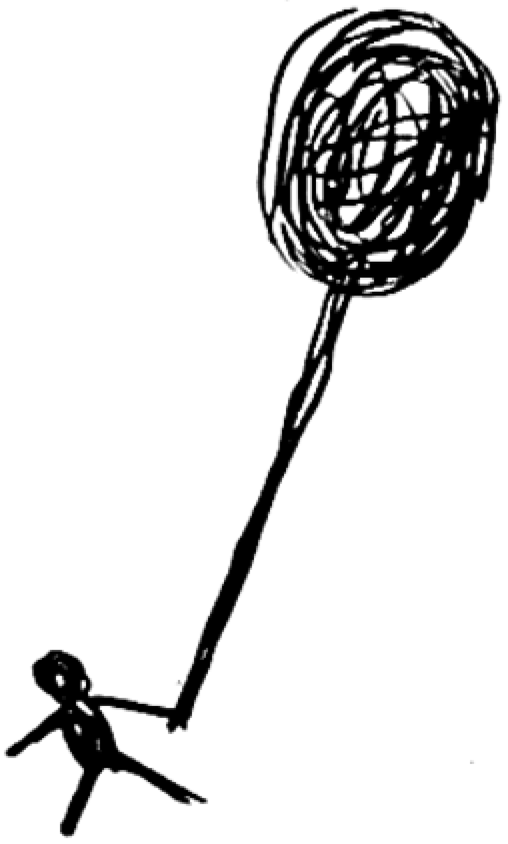
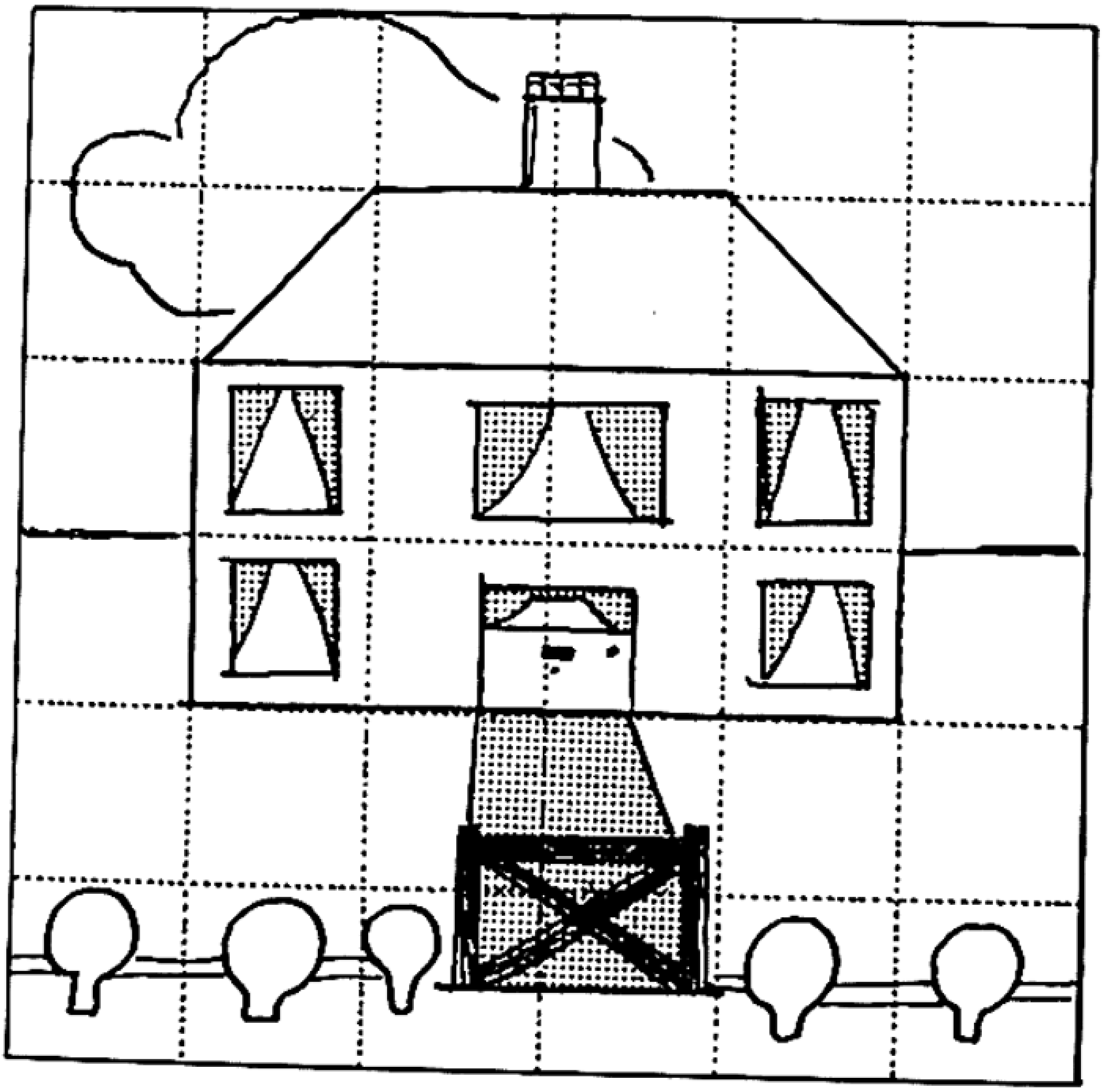


FIG. 45.

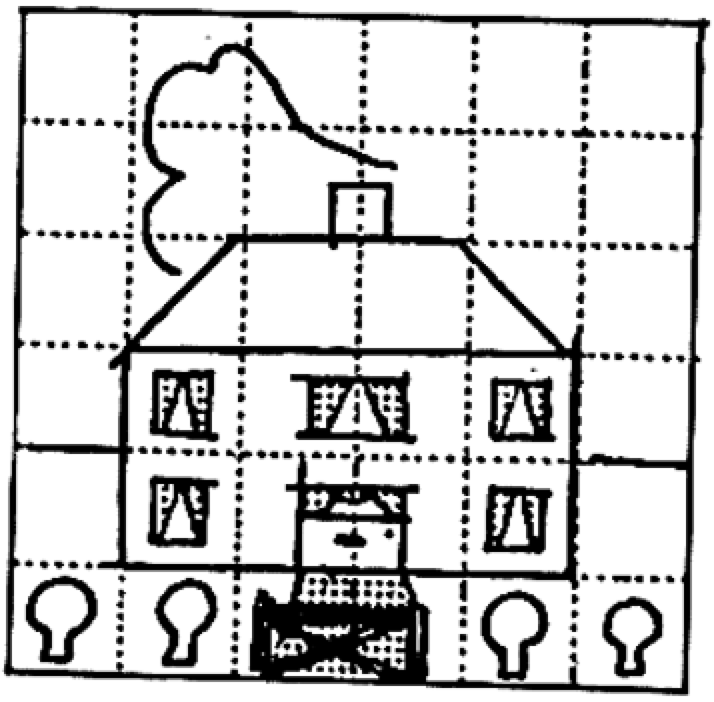
The emotional scale here placed the lollipop conspicuously large and himself a tiny figure in comparison.

Rosemary, at the age of 5, liked to have maps which were partly pictorial drawn for her when she went on walks and drives. She showed early appreciation of shape, for example, in doing jigsaw puzzles.

Pictures, aeroplane photographs, plans, maps, and diagrams are now so common in books, posters, and films that they have become a part of the environment of modern children, who do appear to accept these substitutes for reality at very early ages. Curious drawings with mixed plan and elevation often appear in young children's own drawings; the round pond with people drawn flat radially out from the bank is a recurring example of this type. The ideas of sameness of shape and of a numerical relationship between an object and its picture or model must not be allowed to interfere with artistic expression. Scale drawing belongs more properly with useful activities. It is important, therefore, that first ideas of scale should not only be treated in numerically simple cases but also that they should be harnessed to some important practical purpose. Dissatisfaction with some pictorial record in a project or in nature work might well turn attention to crudely wrong relative sizes. Once a purposeful beginning has been made, quite young juniors will use, for both making and reading drawings, simple scales such as an inch to a foot or easy multiples of linear dimensions.



This is a drawing I have made up myself.



This is the same drawing only half the size.

FIG. 46 (Note mistakes in cloud and path).

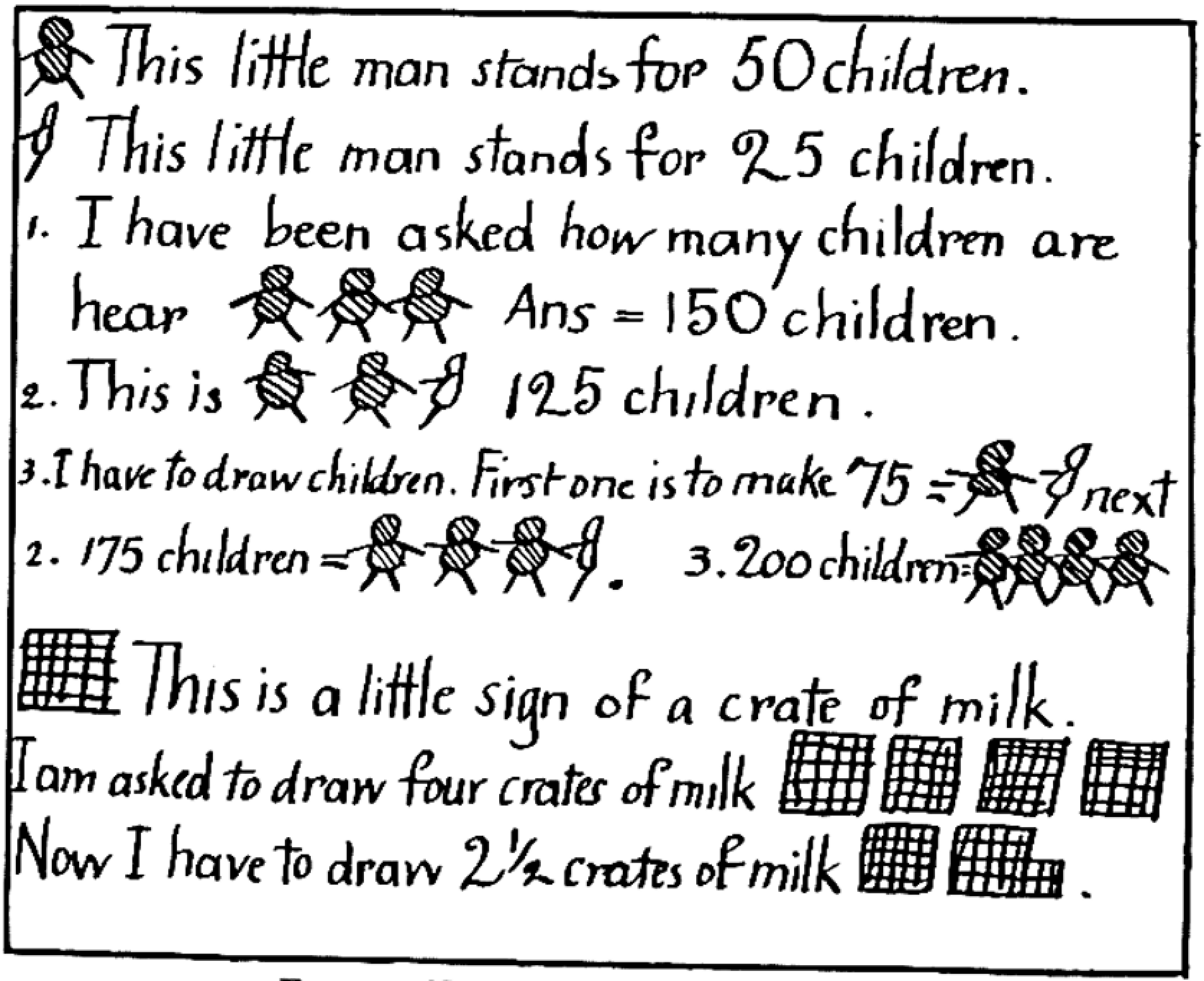
> Some 8-year-old girls did good work in a neighbourhood project in which some of their records and models were roughly to scale. This was not quite their first acquaintance with making and reading scales. On a later occasion, when they visited a farm, they used a given sketch map of it intelligently. The scale of the map framework was given to them. On their return an able member of the class made an enlarged wall map, and then a still larger edition of it in a corner of the classroom floor. On this a farm model was built, and rough distances between buildings were calculated from it by laying down a trail of cotton and then measuring it on the enlarged scale of the framework. Much of this follow-up work was spontaneous.

The measures were, of course, only very rough approximations. The filling in of the enlarged maps was free hand, but it involved careful judgement of positions relative to the framework.

> At a later age, these children were interested in enlargement of pictures on a square network by first enlarging the network, as in Fig. 46. Another group at 10 years of age took to the making of caricatures by enlarging one dimension and not the other.

Older juniors will readily use local maps and a variety of scales. It is wise to keep much of such work well within a known environment, if only because of the difficulty of telling whether spatial appreciation is keeping pace with the numerical work which may be involved. The use of a framework of fixed size for a series of sketch maps of increasing range but including one another brings home without actual numerical work something of the relations between detail, scale, and extent.

Use is often made of diagrams of an explanatory type, which use relative size to exhibit, for instance, the relations between units in weights and measures or money. Although so many measuring instruments use linear scales in measuring quantities which are not lengths, for example, a thermometer or a spring balance of extension type, the representing of something which is not a length by a length may give rise to misconceptions. Diagrams made by the



This little man stands for 50 children.
This little man stands for 25 children.
1. I have been asked how many children are hear Ans = 150 children.
2. This is 125 children.
3. I have to draw children. First one is to make 75 = next
2. 175 children = . 3. 200 children =
This is a little sign of a crate of milk.
I am asked to draw four crates of milk
Now I have to draw 2½ crates of milk .

FIG. 47 (Note mistake in last picture).

children themselves generally begin as pictures of familiar objects, which may in time degenerate into formalized and numerically related picture symbols. Examples of this are given in Fig. 34 (p. 99).

The column diagram of a frequency type does not seem at all difficult when the particular case is well within the children's experience. Chapter V, Figs. 27 *a*, *b*, *c* shows scores recorded with apparatus in this way, and it is there supposed that these records might be made by infants or quite young juniors. Records by children of wet and fine days, of amounts of collections, of attendance, and so on, may well occur in column form at almost any stage of primary school life. So-called isotypes in which a picture-symbol represents a group of units are within junior range. Fig. 47 shows a record of this kind which includes a useful revision of hundreds notation, since two pictures represent 100 children.

Graphical records showing related changes of two quantities might be expected to be outside the normal range of the primary school. Nevertheless, when changes occur in particular circumstances well understood by the children and interesting to them, even infants may enjoy and show grasp of the meaning of a graphical record. An example of this is a record of growth of a plant or flower stalk shown daily by a drawing on a frieze.

> A group of older infants made such a record by moving the plant along daily and making the drawings on boldly squared paper fixed behind the plant. At the weekend two spaces were left as a matter of course, and a 6-year-old girl ran her finger from tip to tip of the Thursday, Friday, Monday sequence of drawings to show that she knew where the tip would come between whiles. On a closer attempt to interpolate she insisted on locating an intermediate height on a line of squared paper rather than in a space, though drawings were generally made in the spaces.

This clear evidence of lack of full appreciation of the meaning of distance along the time line was only in accord with expectation at this very early age, despite the desire to interpolate at an exact position.

> A country boy of 8 years of age, who made a similar record of a growing bean over a long period, showed appreciation not only of continuity and of equal distances for equal times but accompanied his record by comments on the rate of growth, including notes of the conditions when growth was obviously checked.

> An extremely crude graph of this type was achieved by a very backward boy of 10 (working in the same class as the girl who made the diagram for percentages shown in Chapter X, Fig. 38). For this boy the counting of a few squares to record height was an achievement.

Any early use of the graphical symbol should be associated with particular real circumstances well-understood by the children, so that on the one hand the graph is illuminated by its real significance and on the other it records interesting circumstances.

Work connected with scale drawing, the use of maps, and the making of diagrams and graphs must be closely bound up with children's interests as a whole if it is to be really effective. It would, therefore, be unwise to try to map out possibilities in detail. Anything which becomes remote from its real significance at this stage may easily have little point and not be worth including. On the other hand, if interest and purpose are present, numerical difficulties will seldom stand in the way of particular achievements, even when the general processes of arithmetic which would appear to be needed lag behind the particular solutions. One very fertile source of scale drawings is the making indoors of records of measures which have been made by simple means out of doors. Some of these might not be fully quantitative, for example the fixing of the positions found by pacing of points of interest on a winding path only roughly sketched. On the other hand drawings made from the results of pacing the four sides of an irregularly shaped piece of ground might lead to a discussion and to a decision to make some further check on its shape.

It is even more difficult to give any general suggestions for the introduction to other aspects of geometry; the activities in which opportunities occur are widely varied. For instance, an example of parallelism has been given in a context which it would be difficult to reproduce with any spontaneity. No example has been given of making the important decision about what is meant by the distance of a point from a line, but the need for this might occur. Fixing position is another problem which may arise from many points of view and be approached inventively as well as by problem solving, for example, the buried treasure must be buried and recorded as well as discovered. The request for an answer from the child in the back-row-but-one three places from the window wall is a fixing of position by co-ordinates. An example of geometrical experiment arising from a story is given as an example of the range of opportunity.

A class of older juniors had as its serial story a tale of family adventure. They were greatly intrigued by a method described of signalling between two lookout posts. The signals were two pieces of wood, one triangular and the other rectangular, which could be used separately or together, hung in different positions. Codes of this type were worked out in great variety. They involved interesting experiments in hanging shapes up as well as in considering the possibilities of difference in appearance. The code signals had to be listed systematically to simplify reading.

Junior syllabuses generally include some work dealing with the measuring of area. All too often, haste to make the work numerical, and in particular to introduce computation of the areas of rectangles, leads to a slurring over of the important geometrical ideas involved and hence to imperfectly understood rules. The idea of area itself as something measurable and worth measuring does not readily come within the awareness of young children, and the possibility of equal areas with different shapes has to be explored, much as a little child becomes aware by experiment that differently shaped cups may hold the same amount. The notion of boundary surfaces, however, is generally acceptable at a late junior stage, and some qualitative observations will have been made about them, for example, curved, flat, and so on. 'How much surface ?' is not a question likely to arise in precise form ; it might possibly seem important if some large object had to be painted and there was a question of how much paint would be needed to spread a thin layer on the surface.

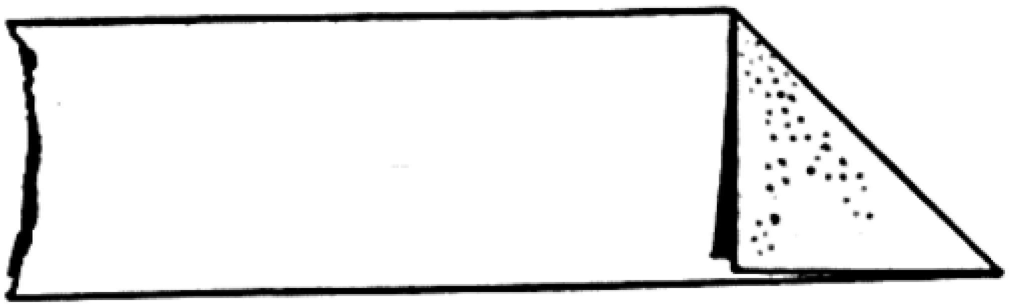


FIG. 48.

A deliberate approach to the comparison of plane areas and to the use of a unit of area was made in a class of 10-year-old

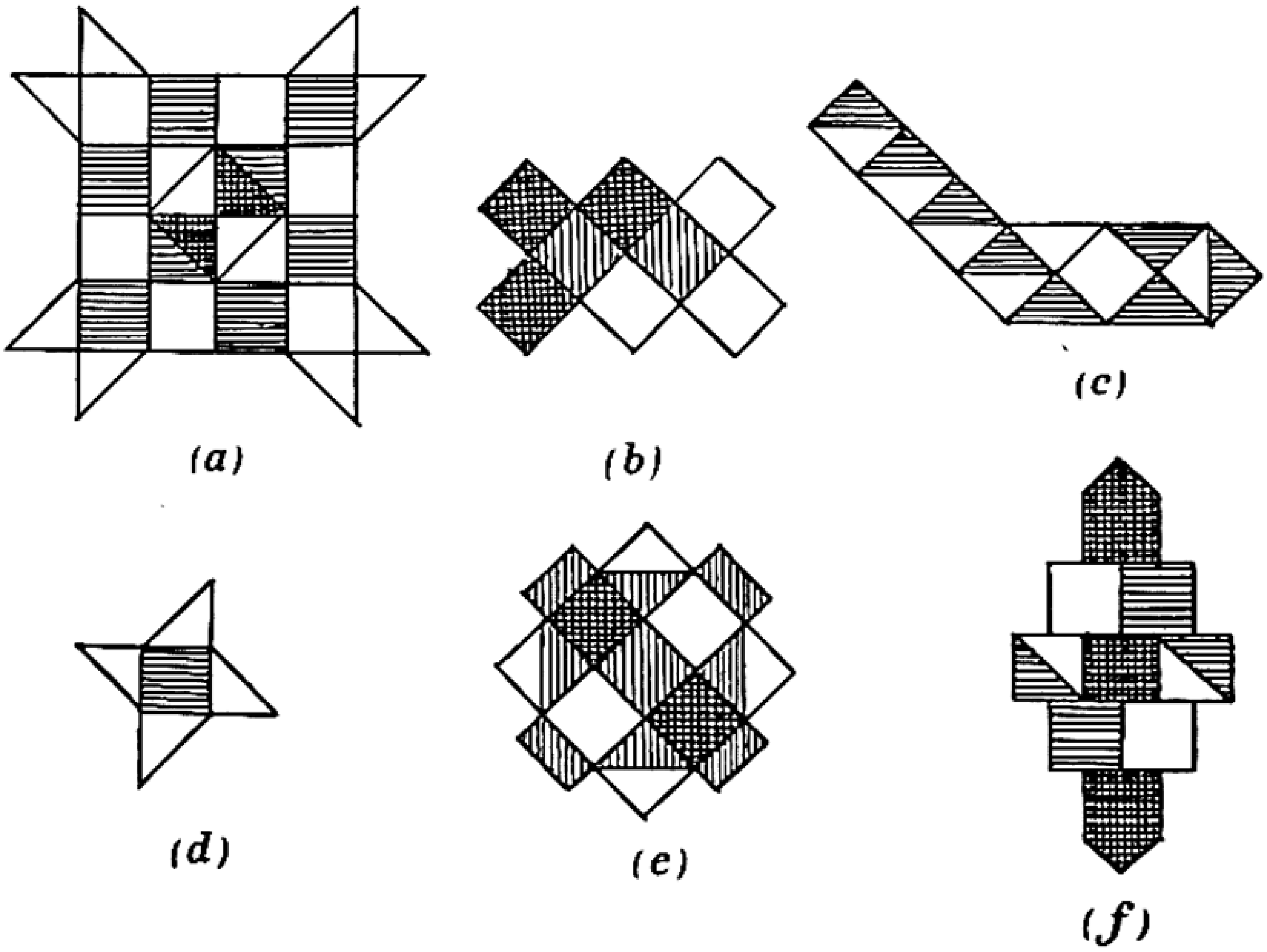
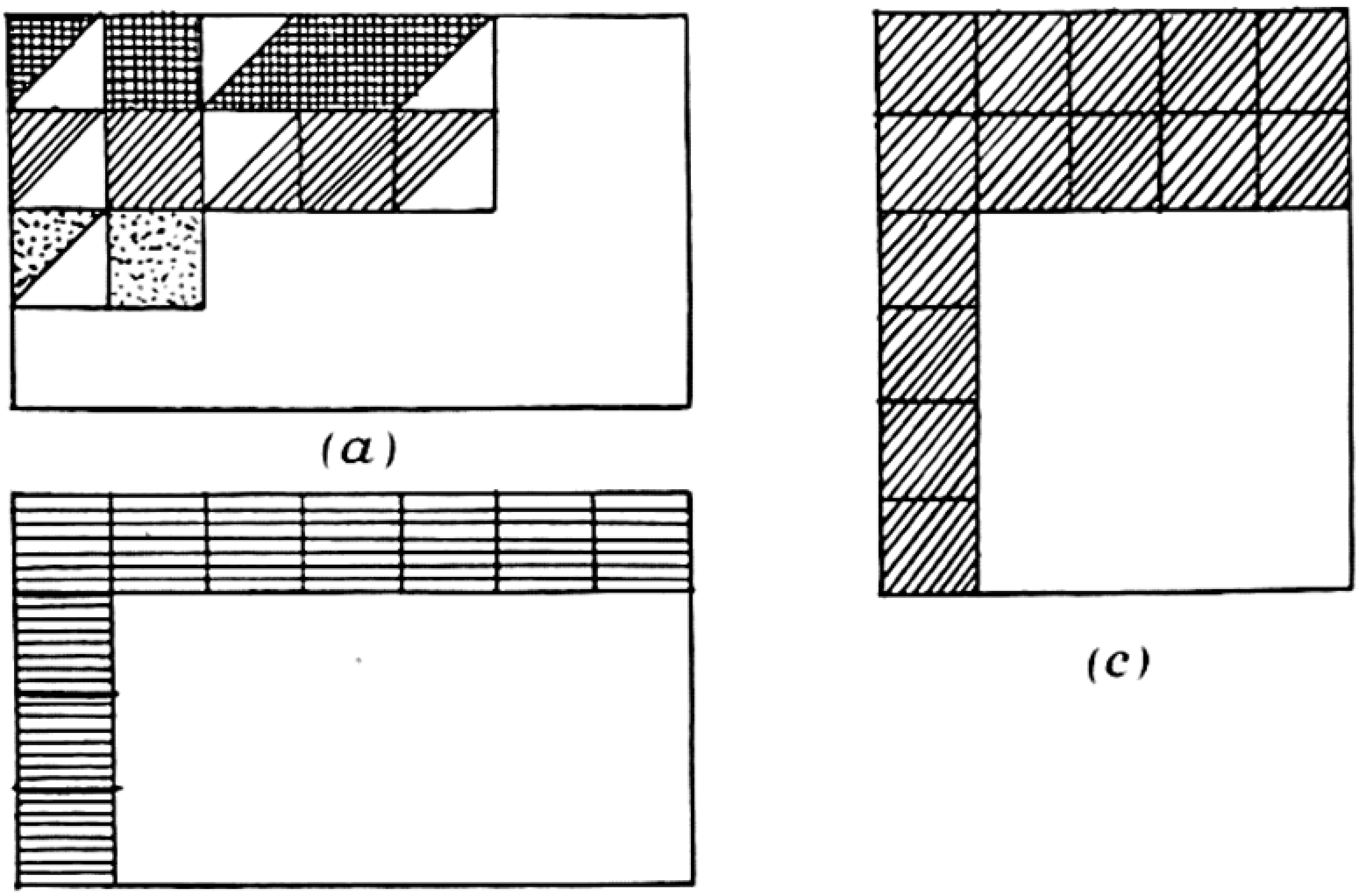


FIG. 49.

children by following up an interest in tessellated pavements and patterns based on squares and half squares.

A few of the patterns made in three colours of cut paper, some made by children and some shown to them ready made, are shown in Fig. 49. The coloured squares were cut from strips an inch wide, and an interesting incidental point arose about how to cut squares. A girl suggested the folding of the strip (as for cutting material on the cross) to obtain a first pattern square (Fig. 48).

The patterns were to be placed in descending order of size. Two boys almost came to blows over (*b*) and (*c*) which are, in fact, equal. The relative sizes of (*e*) and (*f*) were also only determined by a careful count, which shows (*e*) to contain 11 squares of paper and (*f*) 9½ such squares.



(a)

(b)

(c)

FIG. 50.

The transition was then made to comparing two given rectangles of paper which were of different proportions, and which were supposed to be representations of tiled floors. After rather unsuccessful attempts to fill them with pattern had been made, the tiling with a single colour was attempted. This readily led to a computation of their size without complete covering. Fig. 50 shows two such efforts (*b*) and (*c*). In (*c*) two rows of five squares have been applied before the child discovered that the reckoning could be made from a knowledge of the number of squares along each side.

In this investigation one important geometrical notion was assumed, namely that the most convenient shape for a unit is a square. Investigation of the possibilities of other shapes might make a useful revision, or the other possibilities might occur to children with a good experience of patterns covering a surface or of regular shapes, in particular of the regular hexagon and equilateral triangle.

The class above went on to use the formulated rule found in

Fig. 50. The power which their investigation had given them led to alternative methods of computation from an early stage. For example, in reckoning the area of an L-shaped figure (Fig. 51), some children divided it into squares and others completed a rectangle round the L and calculated a difference of rectangles.

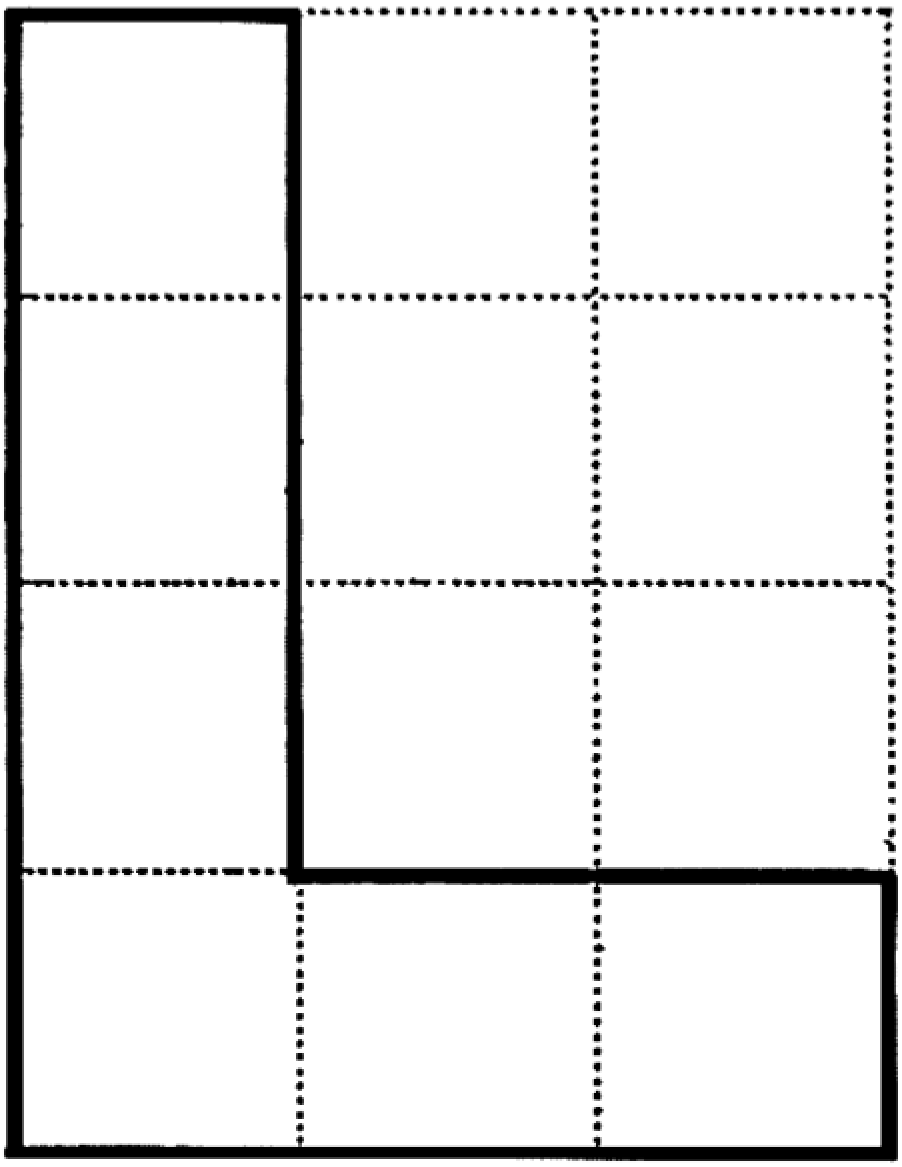


FIG. 51.

Square counting is the fundamental procedure in area measuring. At first, moving cut-out squares is more effective than drawing squares or superimposing a square mesh. The drawing of a square inevitably suggests its outline rather than its area, whereas with cut-out squares the sides are unemphasized boundary edges and the surface is the square. The stage of square counting should not be hurried, and areas of irregular shapes by square counting may well precede the area of the rectangle.

No attempt has been made to discuss any syllabus of geometrical drawing. There are many aspects of it appropriate in the primary school, but it is more important to implant the bold concepts of geometry through practical experience than to develop a scheme of procedures and draughtsmanship. Numerical work connected with drawing and measuring is an important part of number experience and is dealt with in that context. The use of a few basic instruments to make a good job of simple experiences of geometrical form and pattern is important, but how far special constructions and skill in draughtsmanship should be taken depends very much on circumstances and interests.

XII

MISCELLANEOUS OPPORTUNITIES

> The teacher who would cultivate the mathematical imagination of his pupils must begin by cultivating his own.
>
> MARY EVEREST BOOLE
>
> *Preparation of the Child for Science, 1904*

THE interests of the child of junior school age range widely. In this rapidly changing world children may, by taking for granted some new thing, outstrip adults in their adjustments to it. The adult knows that great technical knowledge has gone to build the aeroplane and the television set; but the child accepts as part of life what he finds in his environment and is familiar with such technicalities as have come into ordinary speech, being content with the superficial usefulness which brought them there. Reference to wavelength, for example, in tuning the wireless set carries none of the uneasy feeling of something partly understood which besets those who took to wireless later in life. There is a numbered scale on the set and an adjustment to be made to it in order to find the desired programme. The making of this adjustment calls for a moment's thought in dealing with the numbering of the scale. The reading of the number needed from the programme may bring an inquiry about what the 'm.' stands for after it. This question would allow a very wide range of response and follow up. The one-word answer 'metres' might be given or obtained from the class, and in many circumstances this would wisely be allowed to end the matter. On the other hand, the occasion might be taken as an opportunity for the individual or the class to inquire further. A simple development would be a numerical investigation into different forms of scale and dial, and this might lead to noting scales and dials used for measuring or recording quantities the nature of which might or might not be understood. Another possible line for consideration would be the question of continental measures. This might well be taken as an opportunity for an able group to diverge on a special line of inquiry while the rest of the class went no further than a first contact with the existence of the metric system. A teacher with some technical interest might take interested children a first step behind the scenes, at least into the forms of speech used in differentiating wireless stations, and into some simple consideration of the use of the word 'wave'. Many children acquire technical forms of speech at home and, so long as nothing is said which has to be unlearned, honest inquiry deserves some response; perhaps drawing attention to the other set of numbers marked 'kc/s' by which stations are designated might lead to observations about whether or no they were related to those marked 'm.' Although many of these lines of inquiry are not very fruitful for young children, it has to be remembered that their environment is increasingly full of technical material and that able children of 11 can go surprisingly far, given the practical opportunities along with which there is generally incidental experience bearing on mathematics. The really able junior far outstrips in capacity most modern school pupils, and a knowledgeable teacher can find in almost any expressed interest of such a pupil some opportunity for development.

Miscellaneous opportunities are quite incalculable. The imaginary example described above depends for its usefulness on the occasion, the children, and, perhaps most of all, on the tastes of the teacher. This sort of thing can have no place in a syllabus, but spontaneity of opportunity not only serves to harness energy which might otherwise run to waste or even mischief, but it may also make revision palatable or give entry into new ideas where the deliberate approach would be inappropriate. The opportunities which any new field gives for work with small numbers is one of the benefits most to be valued. Even the ablest cannot have too much practice for their small number accomplishments. The example described has, perhaps, less to offer in this direction than many, but work on a variety of measuring scales, suggested as one possible outcome, generally gives practice in picking out graduations between two tens. This is most valuable for children of below average ability; it may even aid the abler in, for example, the kind of judgement to be made in decimal measurements, or perhaps in enlarging a picture by superimposing a square mesh and then redrawing on an enlarged square network. The difficulty in enlarging is to judge the points at which the picture crosses a line of the network; this might well be improved by practice in decimal partition. There is no guarantee that an interest developed successfully with one group of children will be repeated as successfully with another, and though it is possible to encourage questions by a restaging of fruitful circumstances, the development from them should retain some flexibility.

For teachers mathematically inclined, who also know something of the history, development, and usages of their own accomplishments, opportunities will occur in plenty. For instance, a teacher with any knowledge of statistical usages may find that it illuminates his teaching in all sorts of ways. Instead, for example, of allowing his pupils to accept the rule for finding averages by adding up and dividing by the number of items, he may, by helping them to economical methods of working, give them a clearer conception of what it is they are looking for, and so strengthen the concept of a mean value. One economical method is shown in Fig. 52 *a* in which the common shaded parts represent parts of the items which need not be included in the sum. The sum then consists of the unshaded portions, the total of which must be divided by 4 and then added to the number represented by a shaded column. The other method represented by Fig. 52 *b* is that commonly used: a convenient estimate is made of the average, and the excess or defect from this calculated for the items. The work is shown in two columns, and the difference of the totals taken. The argument then proceeds as for Fig. 52 *a*. So long as the circumstances in which the average is being found are quite clear to the children, and preferably the results of their

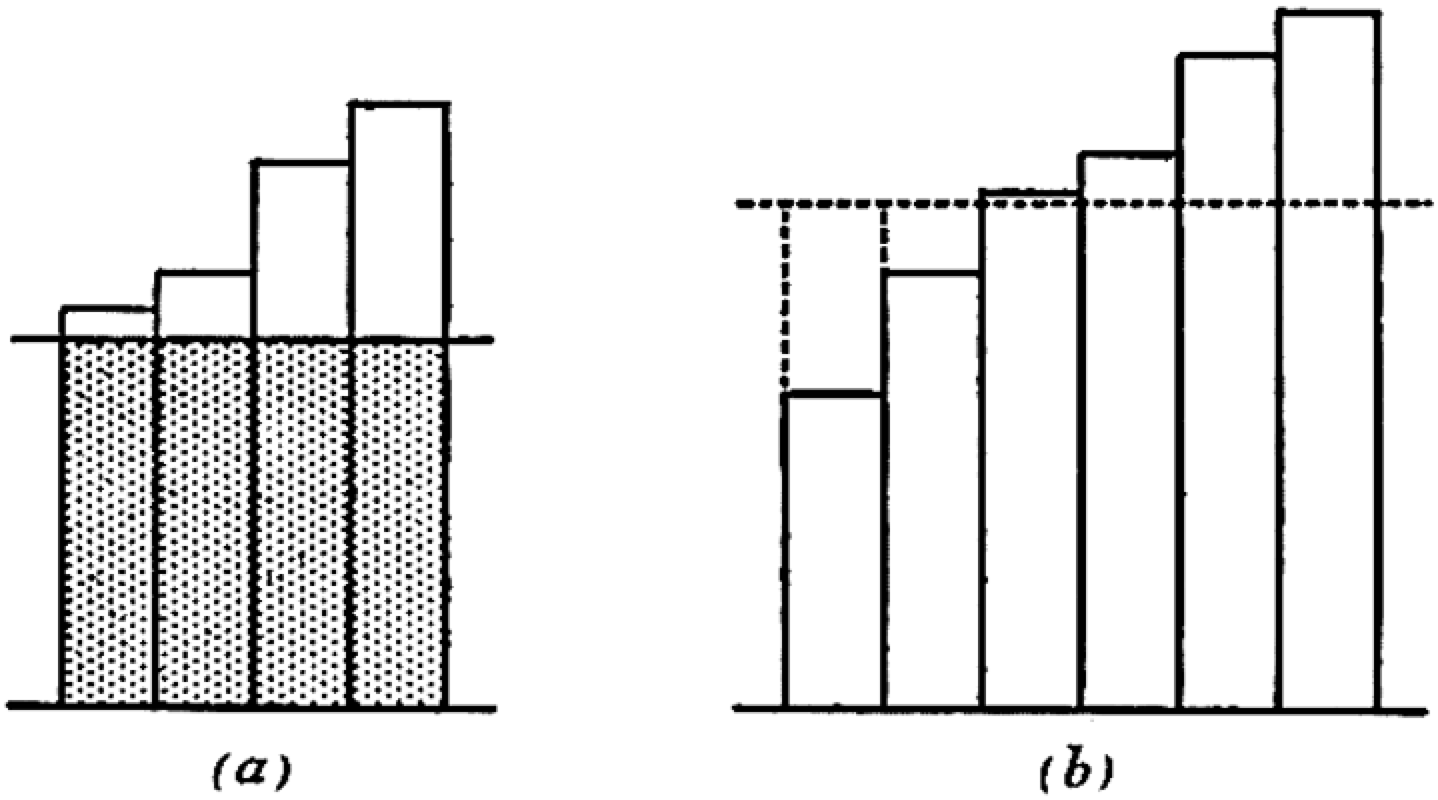


FIG. 52. Average of quantities represented to be found by working from a new zero line chosen, (*a*) less than but near to the smallest quantity, (*b*) near to the estimated mean value.

own practical work, these methods, by keeping the meaning of average clear, present less rather than more difficulty, especially when the quantities concerned lend themselves to diagrammatic representation. Able children could get much valuable practice from this sort of thing, and anyone who attempts averages by the well-known rule should either not do so or be able at least to cope with Fig. 52 *a*. If the quantities in Fig. 52 *a* were 41, 42, 60, 62 a shaded column might be 40 and the excess over 40 of the average is one-quarter of $1+2+20+22$. There are, of course, cases in which neither method is worth the trouble of reckoning differences; these are special cases of Fig. 52 *a* in which the shaded portions disappear.

A quite simple device, often used in statistical practice, is most appropriate for children's use. In collecting data in which a count is required, items are recorded by a stroke until four strokes have been made, and then the fifth stroke binds the group diagonally thus 𝍸; this is called the gate method. Children find it attractive in making an inventory, and it affords excellent practice in counting in fives and tens and in revision of how we write numbers.

> Fig. 53 shows an investigation into the frequency of letters of the alphabet occurring in a newspaper column, made by a second class of 10-year-old children in a large school. The frequency of occurrence of the letters was compared with the simplicity or otherwise of their signal in the Morse Code. The activity of the count and the race to find the most frequent letters in their sample was fascinating to these children. They found this more interesting than the comparison with Morse signals.

If the activity had arisen from an interest in signals, the comparison might have meant more to them. The class was a capable one, but not of high ability. The top class might have been interested in comparing individual samples of frequency, represented graphically, with a class total representation.

It cannot be emphasized too much that such investigations are particular and dependent on interest in the inquiries themselves. Enlightened guidance in the way they are carried out makes them fruitful as a source of mathematical ideas which may only later reach a stage calling for development, and perhaps a new symbolism. For example, it would be a mistake to introduce the use of signed numbers to describe the method of finding averages by working out excess and defect from an estimated mean (Fig. 52 *b*), although by this means all the methods can be linked under one rule. If any development of the idea of directed number were required at this stage it would be more appropriate to seek other instances in which values could be represented on a linear scale on either side of a chosen zero, for example, above and below sea-level, A.D. and B.C., and so on. This work should be carefully treated if it occurs, but in general it should not be sought after till a later age, though in particular instances it presents little difficulty.

> Paul, for instance, who had real mathematical ability and who was favoured in his home background, from the age of 9 helped his older brother with homework. His feeling for the possibilities

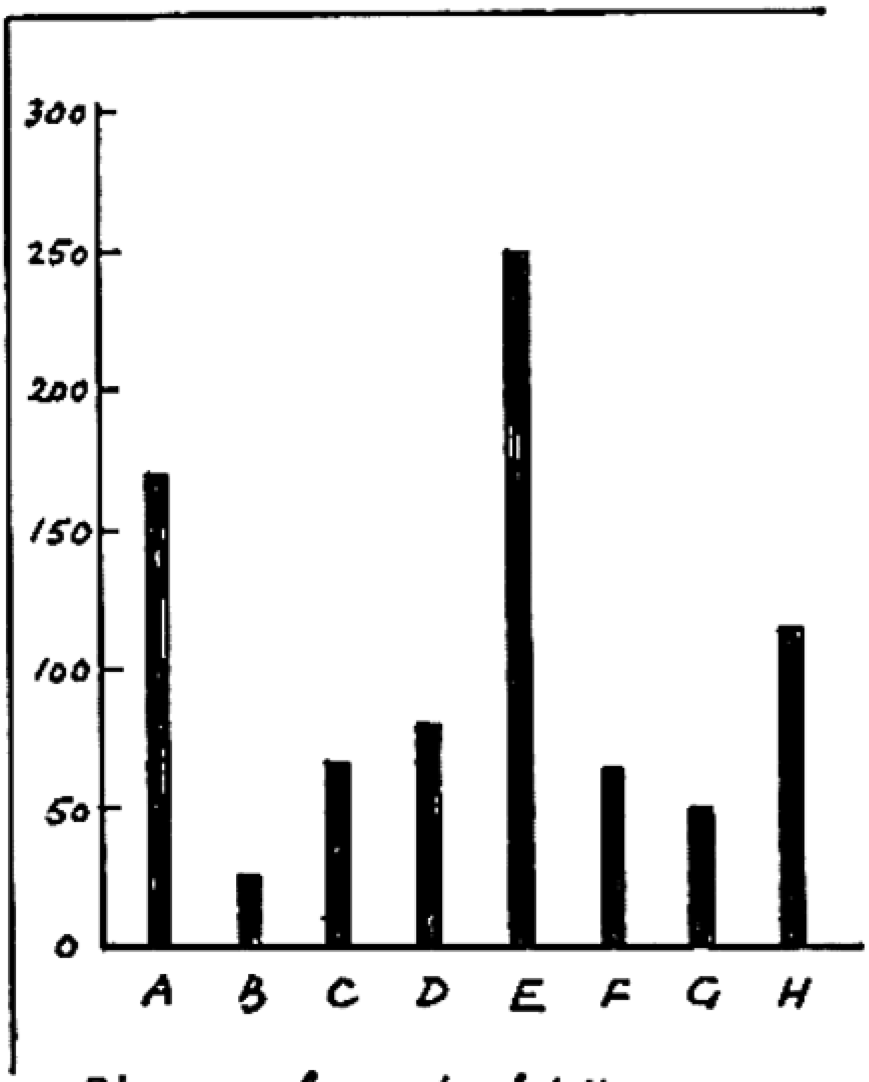
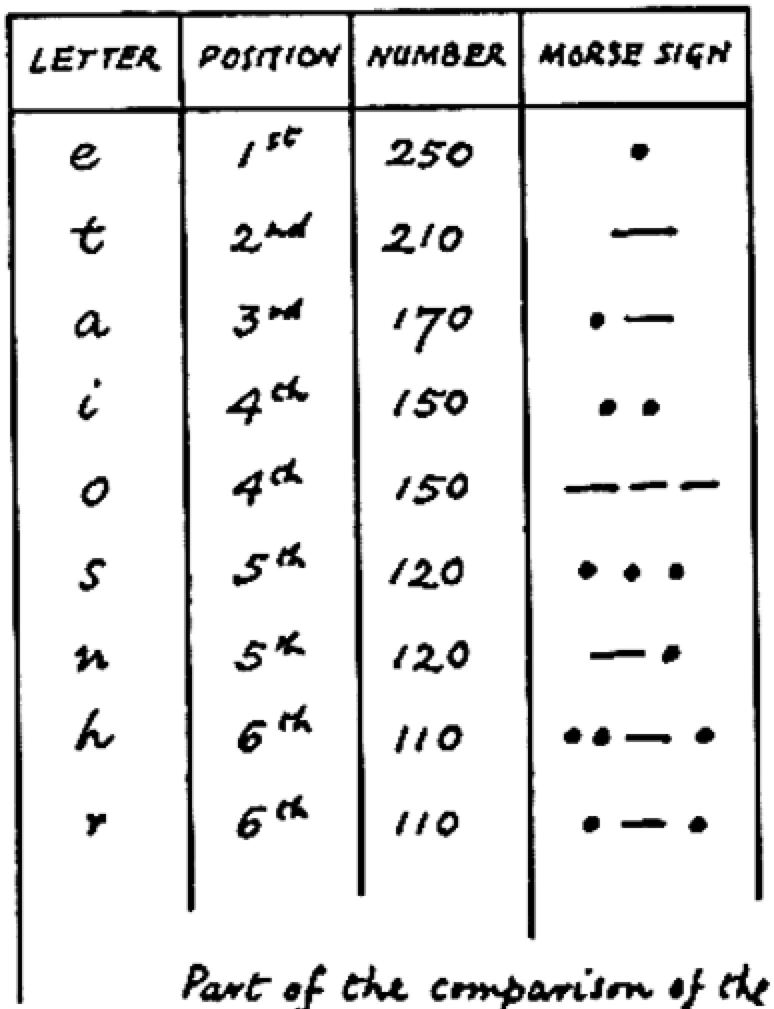
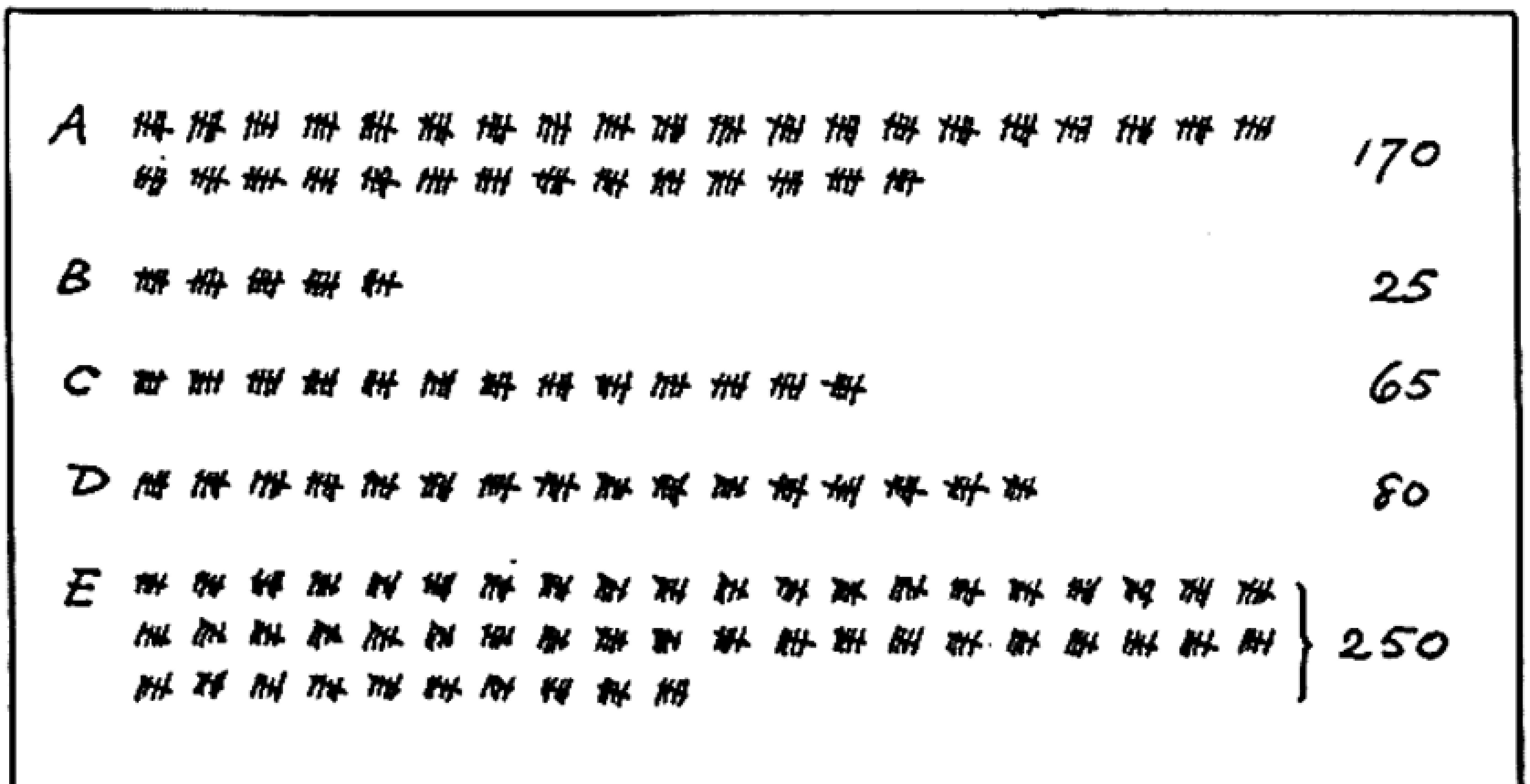


Diagram of count of letters.



LETTER	POSITION	NUMBER	MORSE SIGN
e	1st	250	•
t	2nd	210	—
a	3rd	170	• —
i	4th	150	• •
o	4th	150	— — —
s	5th	120	• • •
n	5th	120	— •
h	6th	110	• • — •
r	6th	110	• — •

Part of the comparison of the frequency of the letters with signs in the Morse Code



Part of a count of the frequency of letters in a newspaper column

FIG. 53. The Alphabet and the Morse Code.

> of mathematical notations was so great that he not only extended his ideas of number to fractions and signed numbers in order to close his number field for division and subtraction, but at 10 he coped with the extraction of square roots and concerned himself with the roots of negative numbers. The computation that was being done by children of his own age was of so little interest to him that for a diversion he would do it and deliberately get it all wrong.

The use of a letter of the alphabet to take the place of a number is a symbolism of very great value in making statements about how calculations have been or are to be performed. Children who have reached the stage of using addition-subtraction and multiplication-division tables readily and appropriately—a stage which is often assumed to have been reached much earlier than, in fact, it is—are ready to say what process will be used in a situation of a particular kind. For instance, to take the most difficult case, that of division: suppose a number of cards of buttons, each of which cost 6*d.*, have different numbers of buttons upon them; a child will find that on a 2-button card each button is worth 3*d.*, on a 3-button card 2*d.*, on a 4-button card 1½*d.*, and so on. If a card is then chosen without being shown, and the children asked how to find what one button on it is worth, older juniors may be expected to reply that 6*d.* must be divided by the number of buttons. The difficulty of such a statement lies in identifying division and not in dealing with the fact that the divisor is only known to the person with the card. As a simpler example, if the price of a ticket is known to be 2*s.* a child who uses multiplication readily will be able to say that the cost for the party is 2*s.* multiplied by the number of people in the party, though the child who still tends to add things up will find statement difficult. The writing of the letter N for the number in the party and saying $2 \times N$ shillings is the cost is easy for the one child and impossible for the other. For able children particularly, a general form of expression might be encouraged when it is appropriate, so as to give formulation plenty of time to become familiar before there is any question of using a formula as a short cut in reckoning. It is suggested that the multiplication sign in $2 \times N$ should be retained, and the shortened form $2N$ left until a much later stage. The most appropriate kind of occasion on which to introduce this generalized form of statement is one in which something has had to be done in a practical way, as well as something reckoned. For example, if a trundle, that is, a wheel and handle, has been used for measuring lengths, the circumference of the wheel must be measured and the number of turns counted. For 20 turns the wheel will move forward $C \times 20$ feet (C being the number of feet in the circumference), or for N turns, $3 \times N$ feet (the circumference being 3 feet). This sort of statement could follow several calculations, and a fuller attempt at explanation.

Practice in saying what has been done, whether in practical work or calculation, is the important accomplishment at this stage; but even this is a fairly mature accomplishment because it brings out into the open again number processes which have become second nature, such as multiplying to find the cost for a party given the cost of 1 ticket. Formulation has been put among miscellaneous opportunities because it is not so much something to be learned as a form of expression to be used when opportunity offers.

Another problem of statement which may occur is that involving ambiguity in the order of operations. It is not very likely to occur except as a computational trick, but the example of the trundle might produce ambiguity if a little piece had to be added on because the wheel would not fit on to the starting or finishing point. Suppose 1 foot was inaccessible at each end, and the wheel 8 feet in circumference, and the number of turns 20; then the statement might be made that the distance was $1+8 \times 20+1$ feet. This statement is ambiguous unless a convention about the order of operations is assumed. It can be made unambiguous by inserting a bracket thus, $1+(8 \times 20)+1$, where a bracket is taken to mean that the contents must be treated as a whole. The important thing about such a situation is that the child should discover for himself that there *is* ambiguity. It is, in fact, one of those circumstances in which one hopes that someone in a group will be so uncertain of what is being done that a mistake will be made which can serve as food for discussion. Interest in the form of a statement cannot be expected to go beyond satisfaction in writing something which is unambiguous and easy to read.

Children do, on the other hand, show from an early age interest in numbers or series of numbers which have marked pattern or unusual shape. It has been pointed out earlier that this interest is not confined to able children, though preoccupation with syllabuses of useful reckoning seldom allows even those teachers who share the children's delight in pattern, puzzles, riddles, and collections of oddities, to stray far into a very rich field of most miscellaneous opportunity. The possibilities range from the pure trick, such as cutting twelve in half to get seven by putting a line through the middle of the Roman twelve, ~~XII~~, to questions of historical interest and mathematical importance. To include in computational practice an occasional surprise may encourage children to be on the look out for numbers of special form or for the implications of a sequence of processes. For instance, the products 27×37, 9×74, 3×185, 6×148 have the results 999, 666, 555, and 888. This is because 3×37 (111) is contained in numbers chosen, though these common factors are not immediately obvious on inspection of the numbers given for multiplication. Again, the type of surprise involving a reversal of quantities or digits is attractive; for instance:

	£	*s.*	*d.*
Write down any sum of money less than £12,	11	15	4
reverse it,	4	15	11
find the difference of these sums.	6	19	5
Reverse again and add.	5	19	6
The result always is	£12	18	11

A reversal surprise for numbers is:

write down any number,	14638
reverse it,	83641
find the difference,	69003
add the digits	18

The result is always a multiple of nine.

A simple geometrical property of a circle which might arouse a good deal of inquiry can be introduced by the device that seven equal coins can be placed so that six of them form a ring round the seventh in contact with it and with the coins on either side in the ring.

The field of possibilities is too wide to illustrate fully, but that children enjoy number puzzles there is no question, even when the reason behind the oddity is beyond them. An example of a device, the explanation of which is a little too formal for juniors, but which has often been tried with children of eight and upwards, is the 'Fool's Rule' for multiplying two numbers between 5 and 10. Call the fingers on each hand beginning from the thumbs by the numbers 6 to 10, and then bring together fingers numbered, for example, 7 and 8 if the product 7 × 8 is to be found. The tens digit of the product is found by counting the number of fingers towards the thumbs including the fingers in contact, and the units digit by taking the product of the numbers of fingers on each hand beyond those in contact. The products 7 × 6 and 6 × 6 add a touch of excitement because one of the necessary tens in the result is produced by saying 4 × 3 or 4 × 4 from the fingers beyond those in contact. Such delights belong essentially to miscellaneous opportunities; it would be a pity to cheapen them by too regular a ration from a textbook, but the teacher who has a good store of his own, which he knows well enough to produce on apt occasions, is in a very strong position.

Historical material is another very valuable asset, probably best used incidentally, though some concentration of interest may yield a good return at times.

A class of girls of 9½ years of age had stories of the early writing of numbers and early methods of reckoning as part of their history lessons. They not only gained much in their ideas of other times and other peoples, but they acquired an unusually detached and mature attitude towards computation. Without being confused by actually learning other methods, they looked on the rules they used as the best of many rather than the only possible methods. Indeed, their attitude resembled an attitude acquired from reaching rules by experiment, although they had in fact been taught authoritatively.

Another class of children of 9 years of age had lost interest and confidence in arithmetic. A course consisting of historical material and of items of interesting information about such things as weights and measures reawakened their interest and their powers. Many quite elementary ideas and processes were revised and analysed with freshness and without the loss of dignity which might have been entailed in direct revision of earlier work. The combination of historical items with the collection of realistic experience about weights and measures was similarly successful.

The collection of items of interest from home and neighbourhood, not only about numbers themselves, as suggested earlier, but about the uses of numbers, is not often exploited as fully as it might be. In a rural area, for example, interesting local measures might come to light, such as a 'square of thatch', which is 100 square feet, or the weight of a pig in 'scores'. The square of thatch might give the difficult first step into area measurement, to be followed by examination both of the square and of the measure.

It cannot be expected that all items or even many such items are of value except as collector's pieces. The device of keeping individual records of anything and everything pleasing to the individual, and a class book of items interesting enough to be worth wider attention, is an easy way both of sifting material and encouraging effort.

Among sources of interesting items must be reckoned the class library and other books. A book on fishing or horses may produce unexpectedly interesting numerical items. The book of birds may have a diagram to show facts about migration. Stories may introduce ideas of distance and direction. The teacher who knows where these may be found has a means at hand of setting the mathematical stage without giving away the secrets of his showmanship. There is, too, the actual exercise of reading something which depends for its meaning on numerical material included in it. The reader of these pages does not need to be told that this is an exercise which differs from a story or even a description.

CONCLUSION: STANDARDS

> It will be found a good rule to make all work which is properly intellectual, all which involves serious thought, either individual or communal; and to reserve competition for those portions of time which are devoted to acquiring skill, accuracy, and speed, by practice in what the pupils already understand perfectly. The stimulus of competition, when applied at an early age to real thought processes, is injurious to nerve power and to scientific insight.
>
> MARY EVEREST BOOLE
>
> *Preparation of the Child for Science, 1904*

STANDARDS are determined partly by the quality and partly by the range and nature of achievement. To test them by answers right and time taken is not an infallible guide to quality, even though accuracy and speed are among the essential characteristics of efficient computation. These two characteristics, accuracy and speed, deserve analysis.

Accuracy in the primary school is of two kinds, accuracy in computation and matters of fact, and accuracy in practical pursuits. Accuracy in computation is absolute; the reckoning is right or wrong, even if there is some doubt about the data. This makes computation either attractive or burdensome according to whether the computer generally gets reckoning right or generally gets it wrong. Matters of definition, such as 3 feet make 1 yard, are also either right or wrong. Much that goes to make the building up of this sort of accuracy is personal and emotional. In particular, it requires confidence; both confident knowledge and confident ignorance, that is knowing when the memory is inadequate and thus when to resort to finding out. Sheer fatigue may shake accuracy, and all sorts of personal factors, by impairing neatness, may bring in errors of misunderstanding from the ill-written page. The building up of confidence and other desirable qualities is very much the business of the teacher; it is not solely a matter of giving practice, though practice may serve either as one means of lingering over an idea until it is clear or as a means of acquiring useful skill. It has not been the purpose of the foregoing chapters to decry practice in its proper place, but rather to suggest that if practice is to be undertaken in order to make what is known truly second nature, there must be a background of preparation and readiness for it. Moreover, accuracy in those matters which should be unquestionably correct is an essential of 'good standards', however simple the range of such accomplishment may be.

Accuracy in practical matters is more difficult to assess, for it is generally a matter of degree and not absolute. In so far as it forms a part of standards of achievement it must be considered in relation to the child's stage of growth of body and mind and not simply in relation to what is undertaken. Ticks and crosses must give way to individual judgements. Poor standards judged by some supposed adult competence may arise when a job is either hard or when it is so easy that it fails to engage interest and effort. For instance, a sloppy mess surrounding practical work with liquids may be inevitable at the earliest stages of experience, but, genuine accidents apart, a red light at others. The main field of practical accuracy to be considered from the mathematical standpoint is accuracy in measuring. It is, indeed, a part of the growth of understanding of what is involved in measuring.[1] Discontent with bits over, with failure to balance and so on, should never be brushed aside. It is a function of measuring activities to encourage such discontents at suitable intervals and to supply a sufficiency of tools, both physical and notational, for example fractional notations, for the child to satisfy himself that he has made an approximation to the measure adequate for his purposes.

---

[1] The word 'measurement' has generally been avoided hitherto so that it may retain its mathematical meaning, and 'measuring' is used to signify the child's activity, which, for example, in measuring length involves gradual acceptance of assumptions about the nature of his tools and of space, as well as increasing appreciation of degrees of approximation.

> Andy, aged $9\frac{1}{2}$, was dissatisfied with the measuring which the class was doing because he thought using twelfths to the nearest twelfth left too much over. He was told to invent a better way. He invented twenty-fourths and gave his answers in twenty-fourths.

To say that measuring has been carefully done is one thing, but to say it is right or wrong is a misuse of terms which might delay a true feeling, however inarticulate, for measurement and approximation. Sources of error are often quite well appreciated by young children.

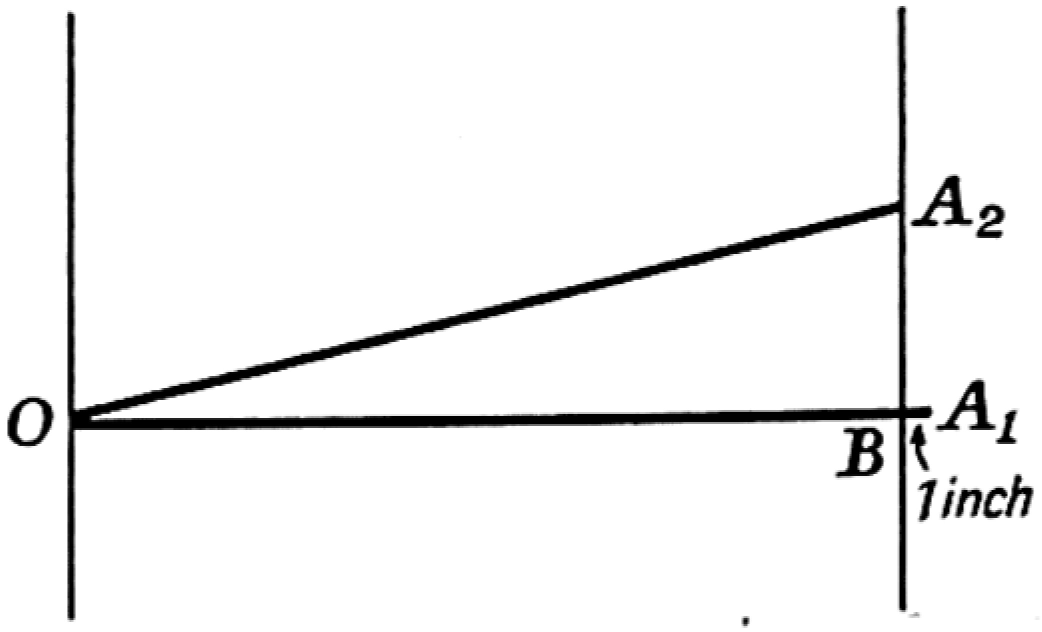


FIG. 54. $OA_1 = OA_2 = 36''$; $BA = 1''$.

> Alec, an able boy of 9, was measuring the length of the corridor. The exercise was not interesting to him and he rather despised his tool, which was a yard-stick. He laid this carelessly along the planking, veering several inches from side to side, though taking some care to keep his end points correctly joined. When questioned about the way he was doing his job, he immediately said that his answer would be too great and suggested he might be reckoning as much as an inch in excess each time. He was asked to place his yard-stick so as to show an excess of one inch. This gave him some trouble, but helped by the planking, a starting line, and a piece of chalk, he eventually achieved a placing one inch short of an advance of 36 inches as in Fig. 54.

This analysis was so near the limit of his powers of dealing with directions and so much more theoretical than practical in its appeal that his interest only just survived to a solution.

> Interest came to life again when he was asked to find better means of measuring a long distance without floor boards to help him. He first tried a 5-foot rod, and then used string to stretch the distance, measuring this afterwards with the yard-stick.

The growth of an interest in accuracy accompanying the use of measuring tools showing fractions of an inch is familiar. Abler juniors may well take an interest in further refinements of measuring tools. Gauges can often be borrowed if they are not a part of equipment. The advance of a screw in a screw gauge is a fascinating device.

To summarize, accuracy in practical matters raises questions of theory as well as of practice, and standards of accuracy must always be judged in relation to the development of the worker and the requirements of the job.

Speed of working is often coupled with accuracy in speaking of standards of computational competence. Accuracy rightly comes first in the partnership, for desirable speed is obtained by superimposing practice and training on the accuracy of confident knowledge. Speed, too, belongs to a utilitarian stage of learning from which the contemplative has been banished. In, for example, the later stages of learning tables—but in the later stages only—this trained athleticism of mind is not out of place. There are, however, dangers in such training which may be likened to the dangers attendant on adult interference with young children's physical prowess and experiment. Strain, anxiety, and competition may make for accidents and damage, physical or mental. Children, however, generally protect themselves from long periods of work which should be done at speed to be worth while by withdrawing their attention from time to time, unless the element of competition is very strong. Lazy habits of work may develop from unwisely set tasks.

Computational accuracy, though it implies some degree of confidence in what has been learned, gives only limited information about standards of achievement. To judge even mechanical written reckoning, it is often necessary to watch children at work as well as to mark answers right and wrong, and to look at the form in which work is set down. So-called problem work set verbally may give a little more information, but the right answer is in itself no indication of what has gone to achieve it, and in any case this highly prepared form of question takes the situation a long way towards solution by the mere fact of verbal statement. The understanding of the statement is, of course, a valuable achievement, but it is a different one from the ability to use mathematical accomplishments in solving problems presented in fact instead of in words.

All these questions are important because of two factors, one internal to a school and the other external to it. Teachers would be more likely to cope with the difficulties of managing children in classes if it were not for the interference of external tests. The problem of dealing with the mathematical development of a large group of young children is complex enough in itself. By 10 years of age differences between children are very great, and they are often exaggerated by uniformity in training which is applied with little regard for the fact that the apt children have acquired foundations for what is attempted and that the average children are still just taking things on trust because they have not acquired enough sanctions from experience.

Sameness of syllabus is sometimes confused with equality of opportunity. The nearest approach to such equality would be to give each individual that for which his mind is ready; this is an ideal which can be faced fairly, even if it must remain far from attainment. It must first be recognized that among children of 10 at the end of the primary stage differences are to be expected which might be described in terms of school organization by saying that (mathematically) some are more mature in mind than most pupils in a modern school, whereas others are still unsure of stages most children concern themselves with as infants. Differing rates of maturation, differing opportunities outside school control, and differing interests cannot be met by attempting to give children an appearance of sameness by instructing them all in the same arithmetical procedures regardless of whether conditions for learning are favourable. Small schools, in which ways of working have to be adapted to differences of age as well as of aptitude, often yield encouraging examples of adaptation to the wide range of children's needs. The following examples are of widely contrasting work from children of 10 who have had the same teachers, but not the same teaching because of difference of aptitude.

> A group of able children of 10 were versatile in solving problems and they showed enterprise in analysing school statistics graphically. The following are two of their descriptions of the solution of the problem, 'A coil of wire contains one mile. If a man uses on an average 110 feet to wire a house for electricity, how many houses can he wire with one coil?'. First solution: 'I did this sum by remembering that there were 220 yards in 1 furlong. But as they are feet not yards I multiplied by 3 and divided by 110 to see how many houses could have wire in a furlong. Then I multiplied the answer by 8 because there are 8 furlongs in 1 mile.' Second solution: 'First I multiplied 1,760 by 3 to find how many feet in a mile. Then I divided them by 110 because he used 110 feet per house'; and so on, to other statements of how units and numbers were dealt with.
>
> At the other end of the ability scale a boy of 10 was suitably employed, though he had confidence in little beyond counting. His graph of a growing bean is mentioned in Chapter XI. A girl of just 10 was illustrating shopping sums, involving the expenditure of a few pence on toys for a Christmas tree, with chalk drawings. She also drew 15 ducks and wrote: 'There are 15 ducks. These have 30 legs'.

It is easier to describe the very slow learner on the one hand and on the other the initiative and powers of expression of the markedly able, than it is to give any picture of the children between, who show their own lively responses to demands made on them when those demands are suited to their stage mathematically and in line with their interests and experience. Teachers who have some depth of knowledge of their children and their needs find their own ways of meeting the most urgent claims for differences in treatment, provided that external influences are not too strong for them.

Extremes of backwardness are often tackled more sympathetically than the subtler differences between the able and the average. The latter often pass so rapidly from stage to stage of mere computation that they do not know what the rules are for and miss all sorts of details of readiness discussed fully in other chapters. In all probability most average children could, when leaving a primary school, make use of small number fairly competently and would have good practical knowledge of weights and measures if they were not hustled through these vital stages. Such rules as they learned would have more permanence and meaning for them. Most important of all, however, would be their attitudes to learning and the extent of the mental development which accompanies ways of learning less dependent on mechanical practice and authority.

Able children, too, sometimes work from rule to rule at a pace which leaves too little opportunity for experiment and initiative; but they learn more quickly from experience and so are less dependent on the practical opportunities offered in the schoolroom. They run less risk of being out of touch with the meaning of what they are doing. Complaints about inadequacies in 'fundamentals' are not, however, confined to the 75 per cent. of children judged to be less able. It is no easy matter to extend able pupils in ways best suited to their abilities. Frequently they are occupied by being given more practice with larger numbers in the formal processes. Sometimes they practise bad habits of mental reckoning as well as good and, if they are uninterested, their work passes muster without engaging adequate effort. If attitude to learning be taken as a criterion of effective mathematical education, able children often show much less than their best; initiative sometimes dies very young. For this, the school of thought which says 'Give us the tools and we can deal with their use' is often to blame. The whole business of forging, sharpening, and using tools of thought is a unity, and at every stage there is something to be explored, something to be practised, and something, however little, fully ready for use in any new field. To cut out exploration and to limit use is instruction, not education.

Finally, something must be said about the vexed question of external testing. A test at 11 years of age affects the majority of children and often affects them from infancy. It is unfortunate for mathematical education that certain aspects of the subject lend themselves to formal testing. It is not unnatural in the circumstances that these particular aspects should be stressed at the expense of others, and that even the less able should race as far as they can along a narrow track. It takes much faith on the part of teachers to follow a balanced approach, even for children under 9, especially in those areas affected by entrance examinations to independent schools which sometimes test pupils for entry at 8 years of age or earlier. The mathematically imaginative child often suffers a severe check by his failure to cope with premature formalities. Even at 11, examinations often include items which might well be left to later years for most children were it not for the handicap which lack of acquaintance with some formal trick or new notation would place on the chances of their going on to the school of their choice. Internal tests between departments of a school sometimes err in the same way, but the cost to the child of disregarding an arbitrary syllabus is not so great.

One of the most unfortunate results of using the examination yard-stick on young children is the effect which it inevitably has on the sense of values of the teaching profession. It has in the past not only given particular aspects of mathematics undue prominence in teachers' eyes, but it has segregated the subject from its fellows, and dissociated it from the growth of the whole child.

PRINTED IN
GREAT BRITAIN
AT THE
UNIVERSITY PRESS
OXFORD
BY
CHARLES BATEY
PRINTER
TO THE
UNIVERSITY

ALLAMA IQBAL LIBRARY
13093

JAMMU & KASHMIR UNIVERSITY
LIBRARY
No 13093..
Date 22.-8-.55
SRINAGAR